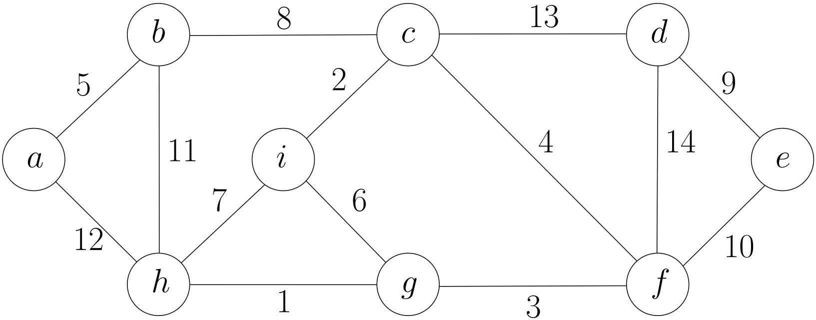
Greedy Algorithm

MST-Greedy(G, w)

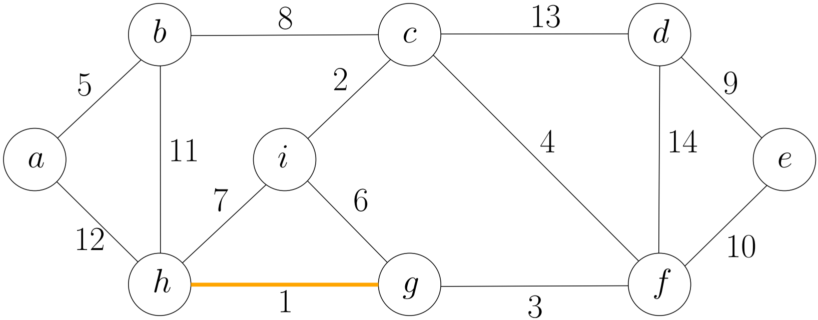
$F \leftarrow \emptyset$
sort edges in E in non-decreasing order of weights w
for each edge (u, v) in the order **do**
    if u and v are not connected by a path of edges in F **then**
        $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

Kruskal's Algorithm: Example



Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

Kruskal's Algorithm: Example



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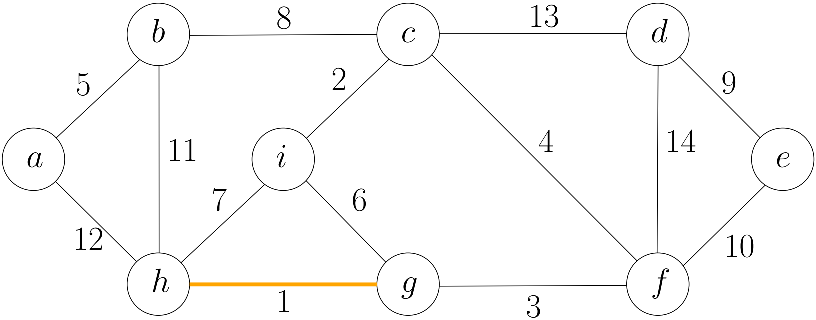

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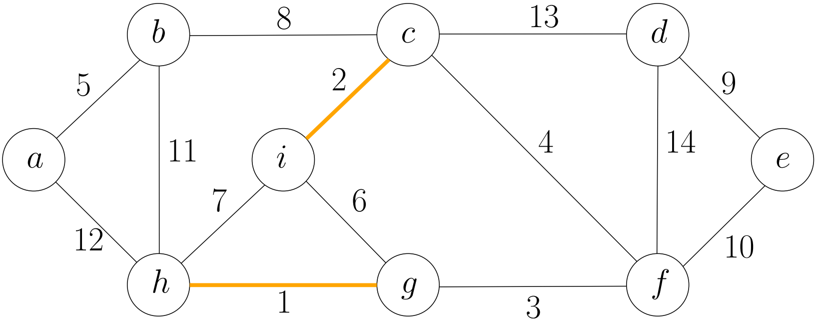

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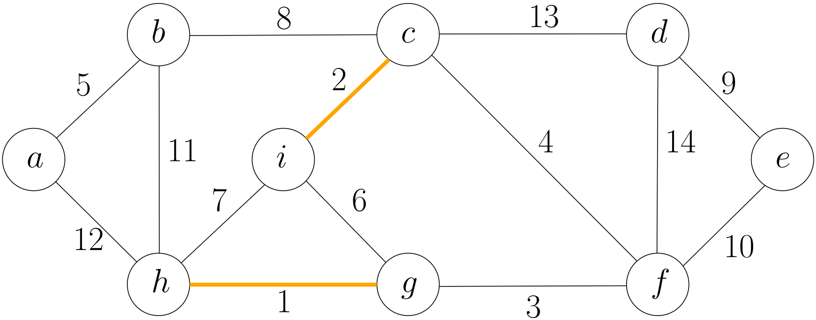

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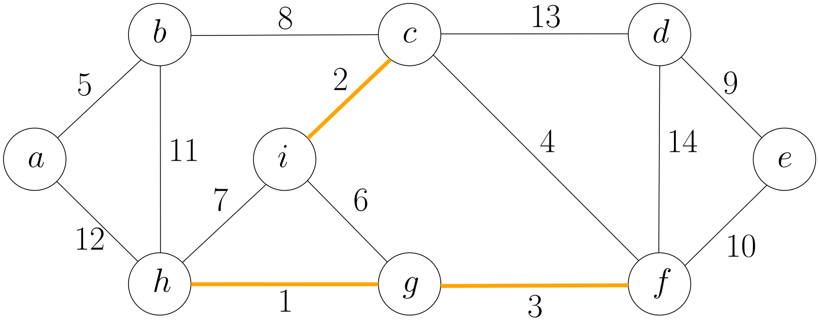

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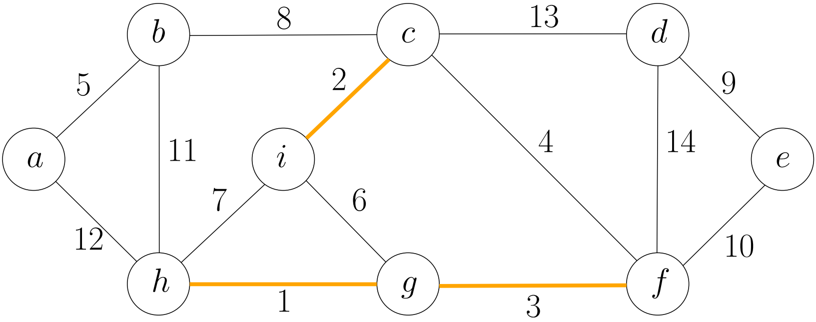

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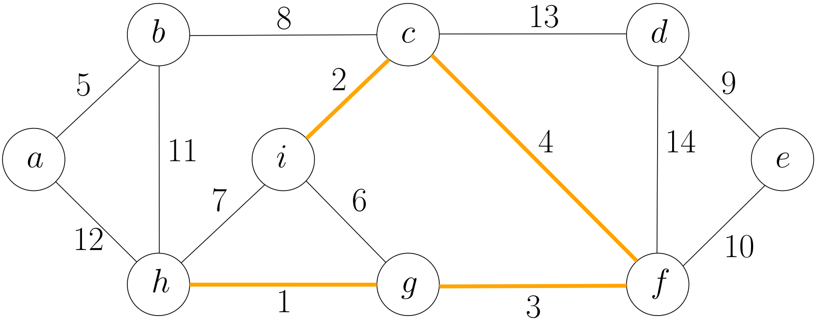

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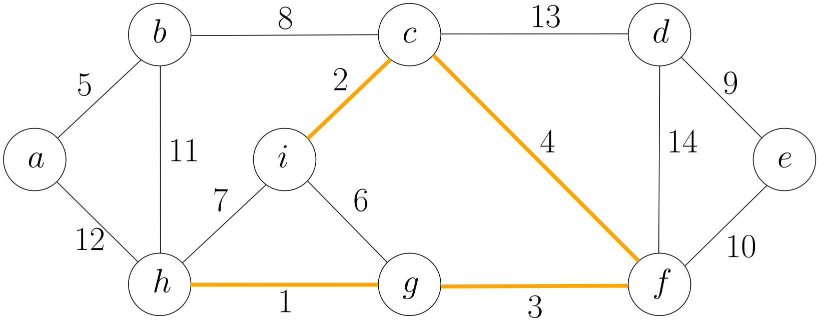

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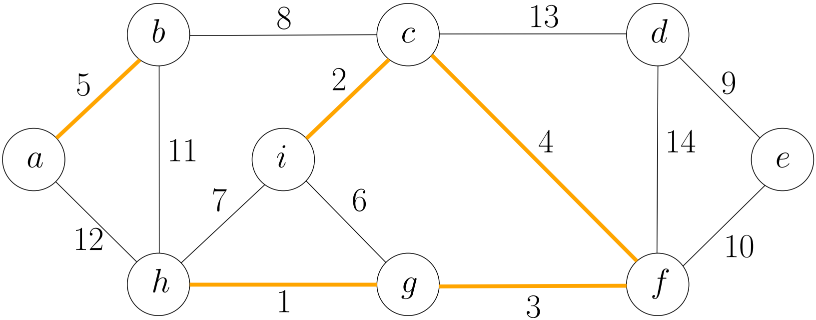

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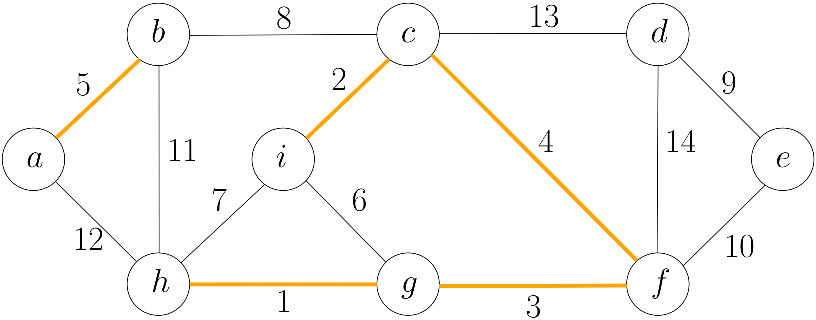

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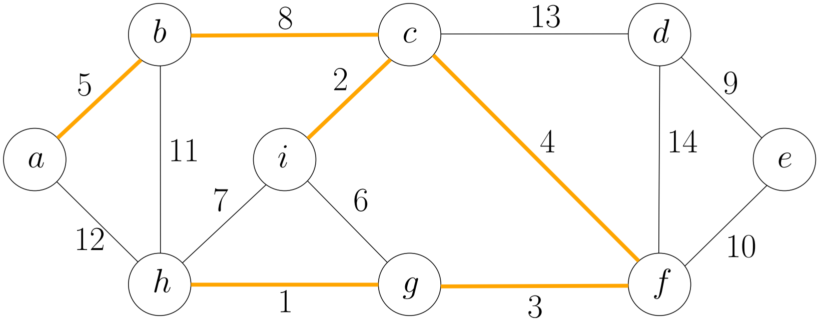

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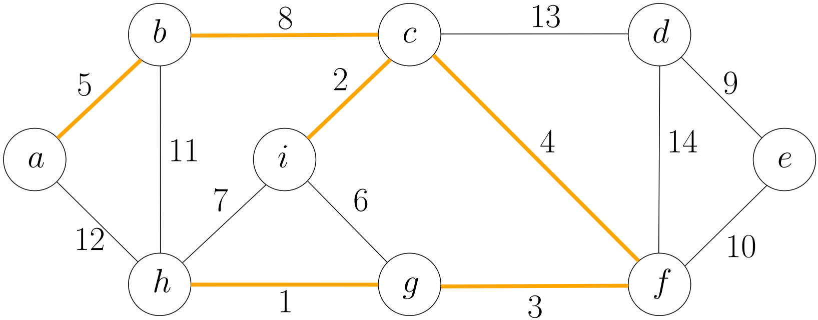

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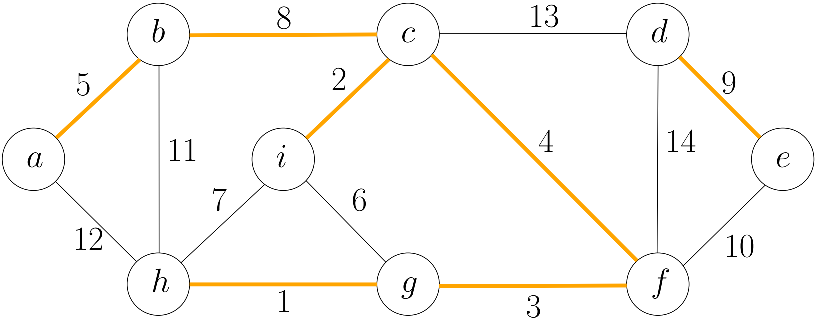

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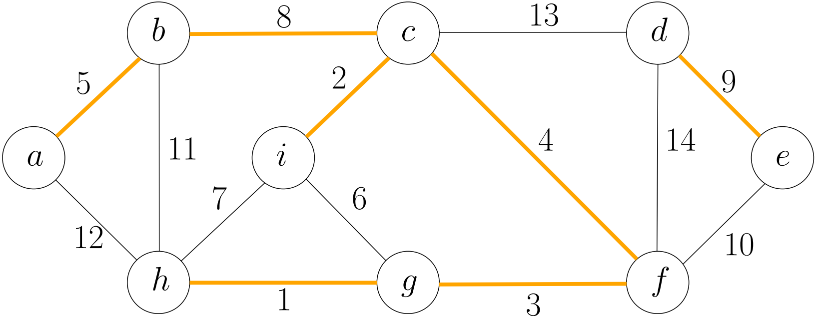

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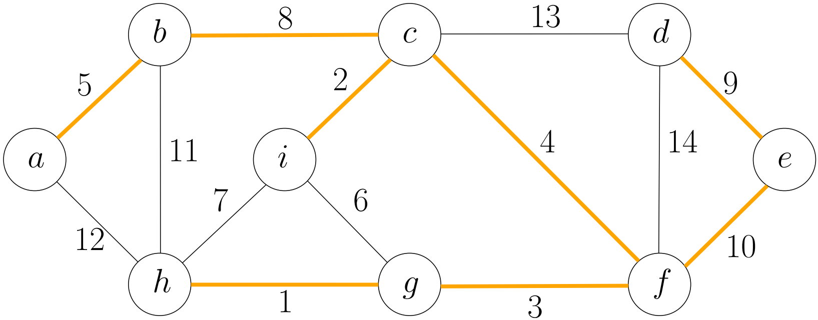

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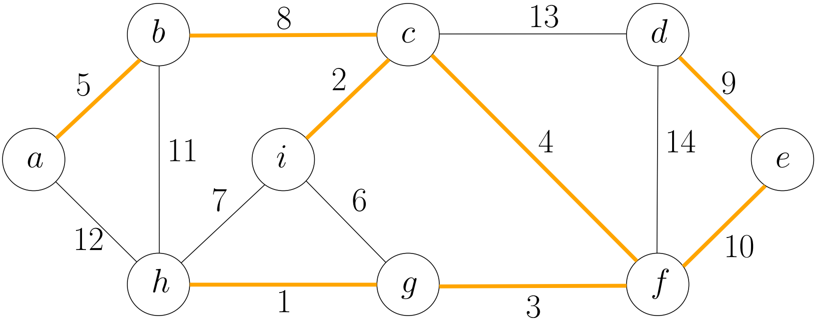


Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$

$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order **do**

  $S_u \leftarrow$ the set in $\mathcal{S}$ containing u

  $S_v \leftarrow$ the set in $\mathcal{S}$ containing v

  if $S_u \neq S_v$ **then**

    $F \leftarrow F \cup \{(u, v)\}$

    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$

return (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

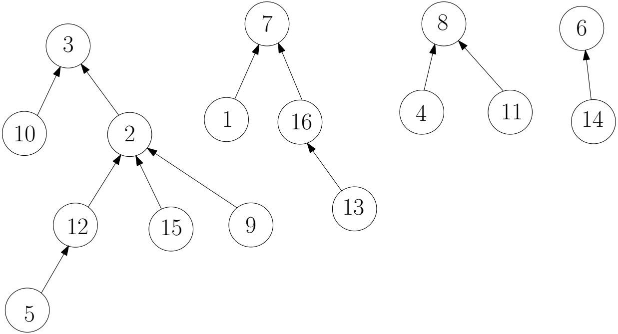
1: $F \leftarrow \emptyset$
2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of E in non-decreasing order of weights w
4: **for** each edge $(u, v) \in E$ in the order **do**
5:     $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6:     $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7:     **if** $S_u \neq S_v$ **then**
8:         $F \leftarrow F \cup \{(u, v)\}$
9:         $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: **return** (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

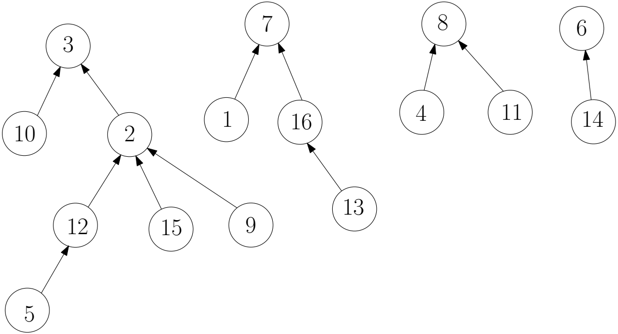
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] = \perp$ if i is a root).

Union-Find Data Structure



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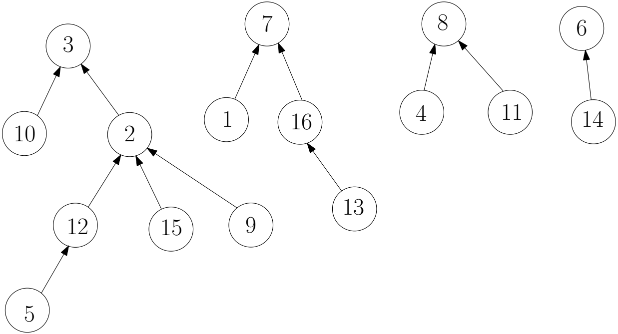

- Q: how can we check if u and v are in the same set?

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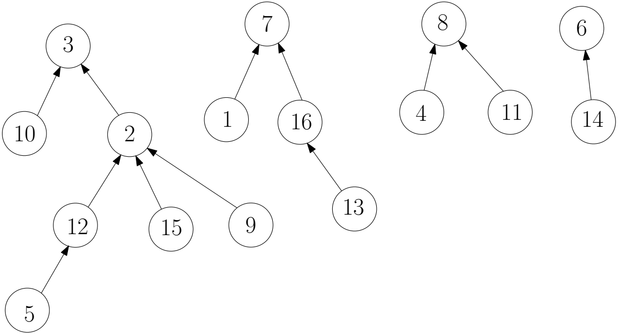

- Q: how can we check if u and v are in the same set?
- A: Check if $\text{root}(u) = \text{root}(v)$.

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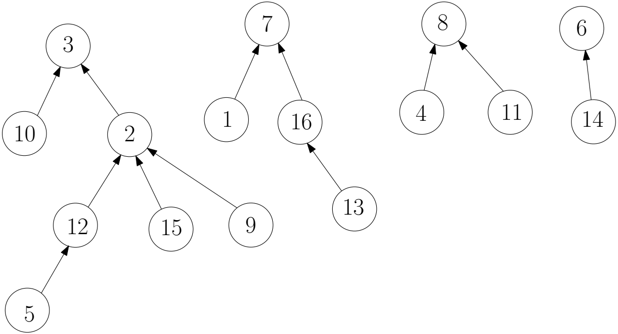
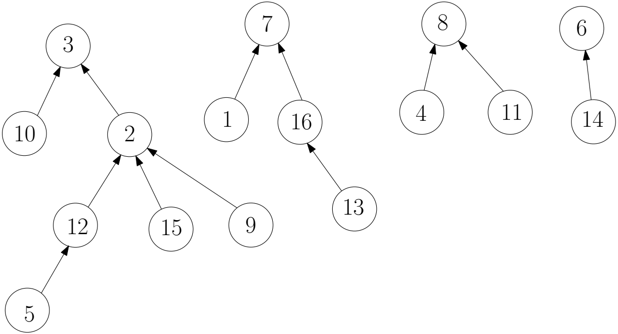


- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
- root(u): the root of the tree containing u

Union-Find Data Structure



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- A: Check if root$(u) =$ root(v).
- root(u): the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

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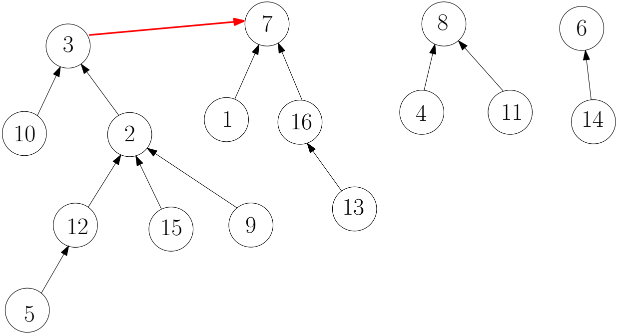

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Union-Find Data Structure

root(v)

if $par[v] = \perp$ **then**
    return v
else
    return root($par[v]$)

Union-Find Data Structure

root(v)

1: **if** $par[v] = \perp$ **then**
2:     **return** v
3: **else**
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- Problem: the tree might too deep; running time might be large

Union-Find Data Structure

root(v)

```
if par[v] = ⊥ then
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- Improvement: all vertices in the path directly point to the root, saving time in the future.

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root(v)

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if par[v] = ⊥ then
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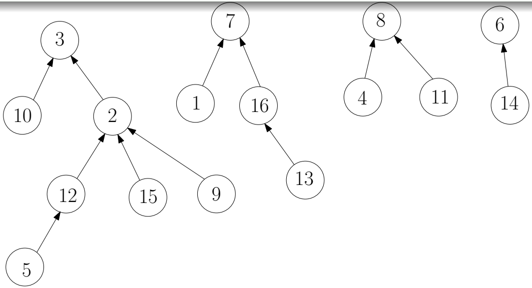

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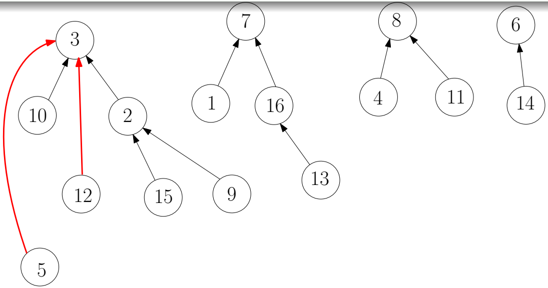

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MST-Kruskal(G, w)

$F \leftarrow \emptyset$
for every $v \in V$ **do**: $par[v] \leftarrow \bot$
sort the edges of E in non-decreasing order of weights w
for each edge $(u, v) \in E$ in the order **do**
    $u' \leftarrow$ root(u)
    $v' \leftarrow$ root(v)
    if $u' \neq v'$ **then**
        $F \leftarrow F \cup \{(u, v)\}$
        $par[u'] \leftarrow v'$
return (V, F)

MST-Kruskal(G, w)

1: $F \leftarrow \emptyset$
2: **for** every $v \in V$ **do**: $par[v] \leftarrow \bot$
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- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

MST-Kruskal(G, w)

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2: **for** every $v \in V$ **do**: $par[v] \leftarrow \perp$
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4: **for** each edge $(u, v) \in E$ in the order **do**
5: $\quad u' \leftarrow \text{root}(u)$
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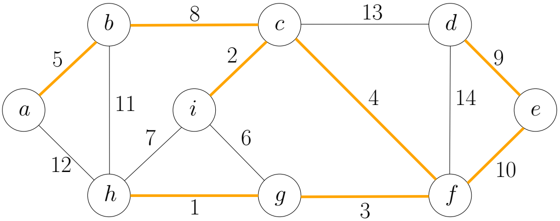
- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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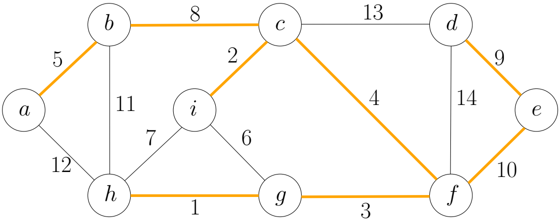

- (i, g) is not in the MST because of cycle (i, c, f, g)

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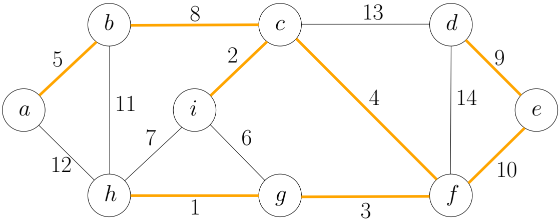

- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree

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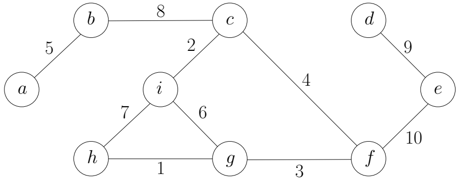

Q: Which edge can be safely excluded from the MST?

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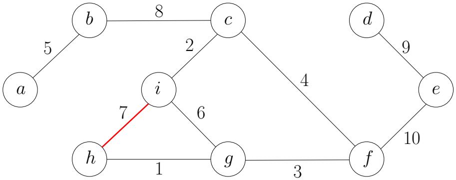

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A: The heaviest non-bridge edge.

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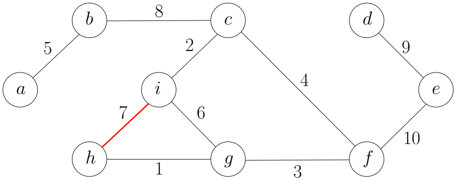


Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

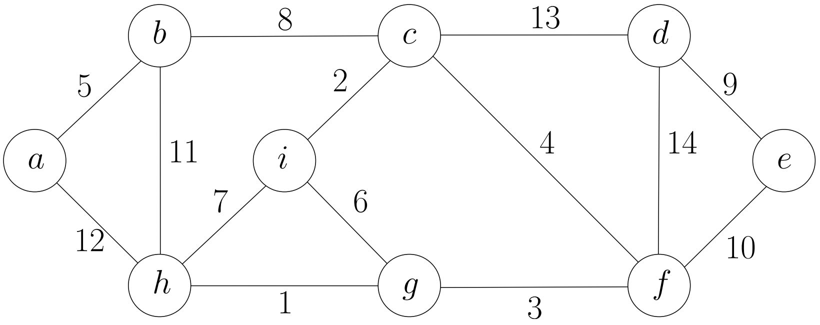
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

```
F ← E
sort E in non-increasing order of weights
for every e in this order do
    if (V, F \ {e}) is connected then
        F ← F \ {e}
return (V, F)
```

Reverse Kruskal's Algorithm: Example



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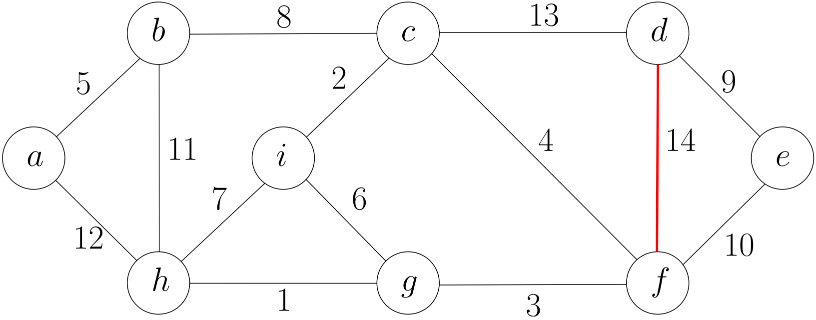

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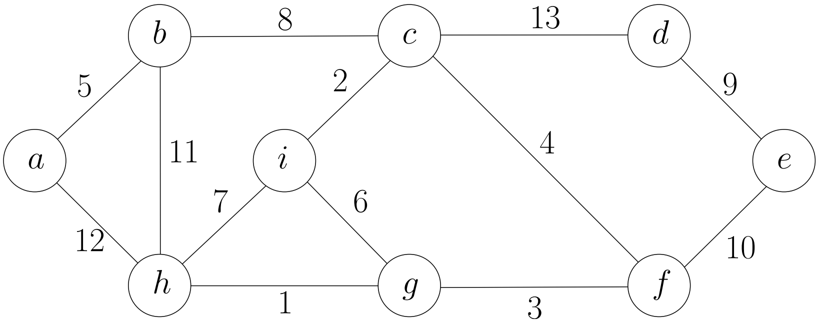

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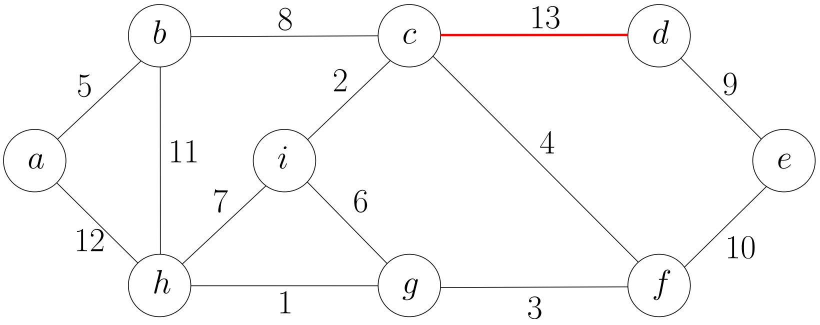

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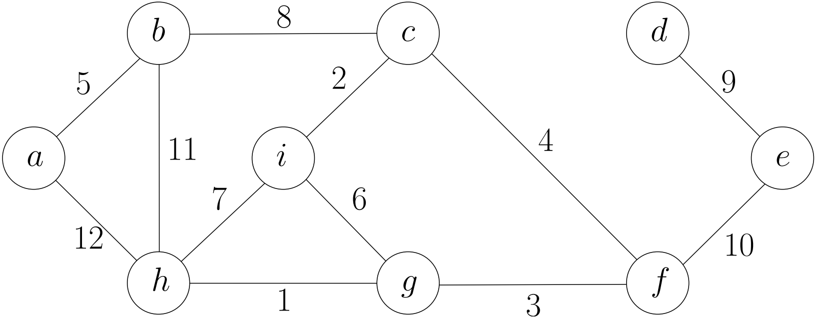

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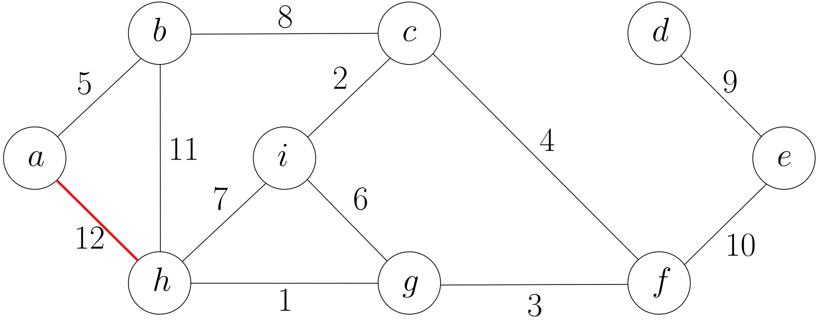

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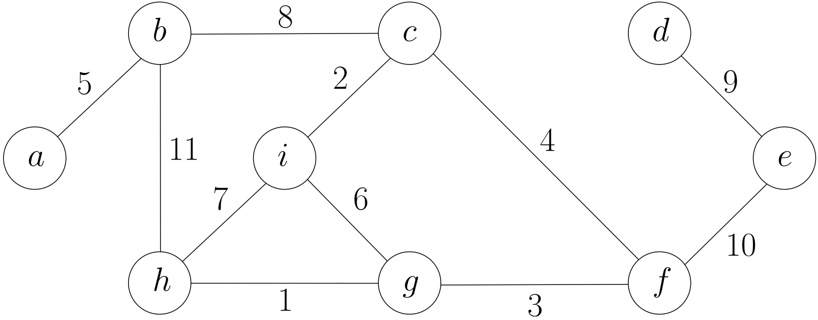

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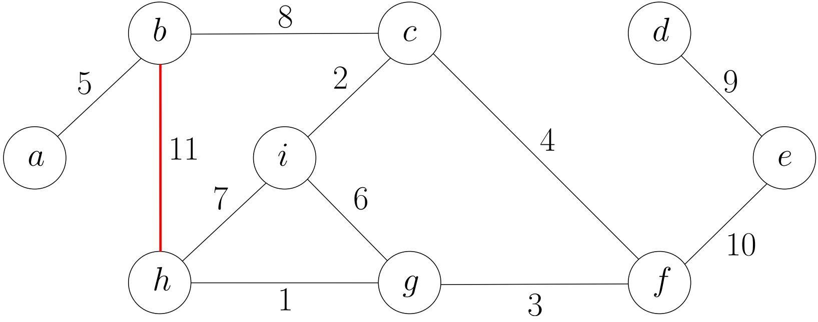

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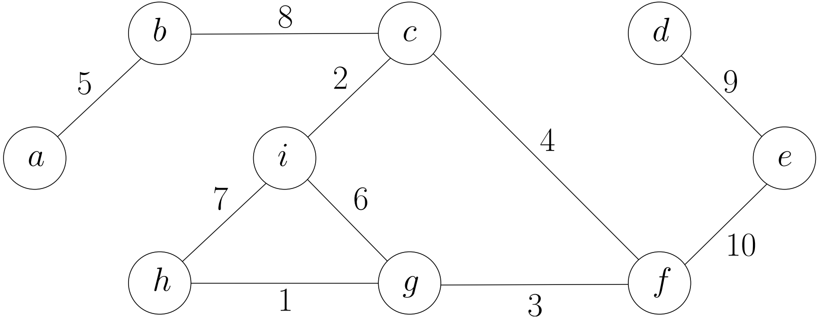

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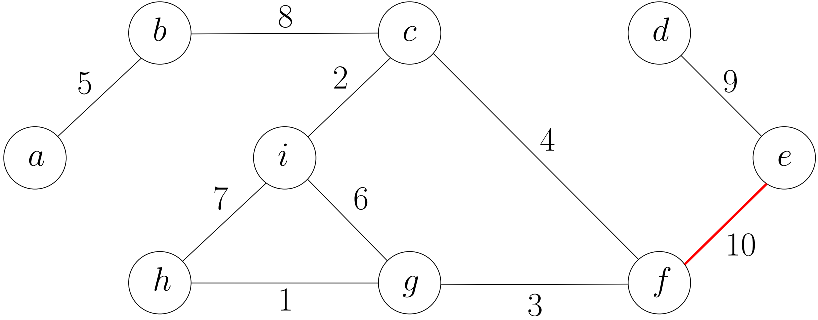
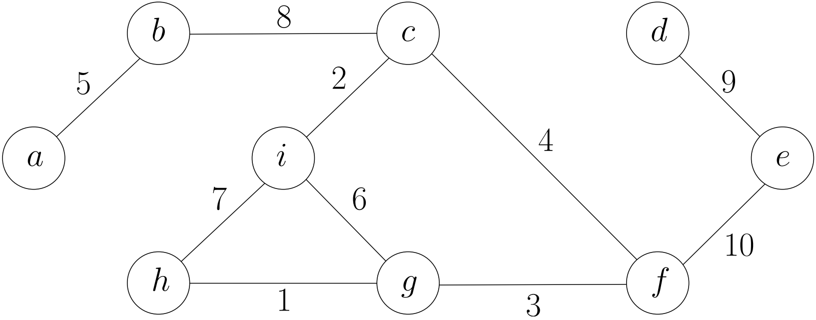
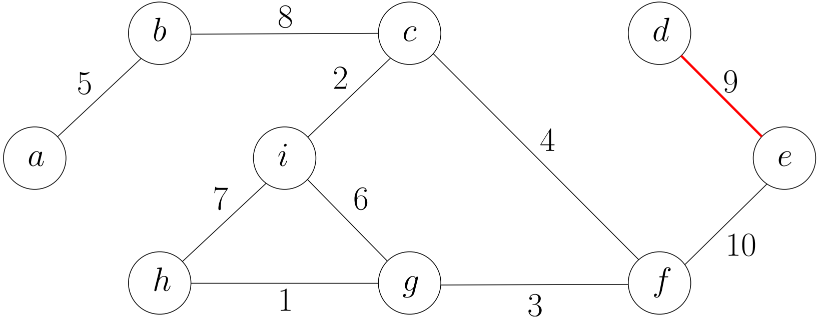


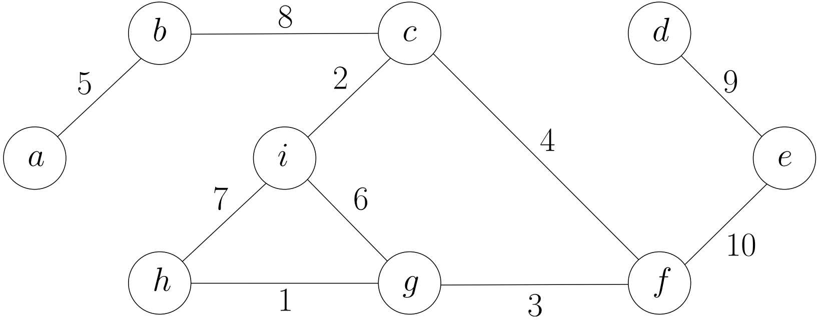
Reverse Kruskal's Algorithm: Example



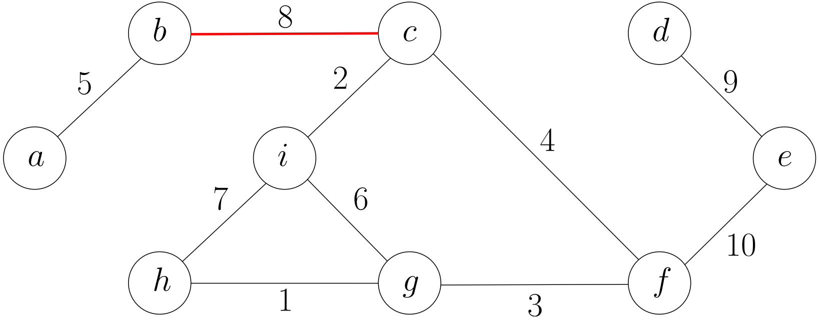
Reverse Kruskal's Algorithm: Example



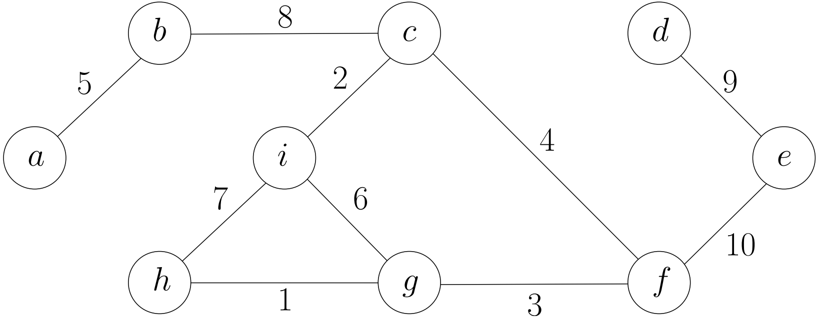
Reverse Kruskal's Algorithm: Example



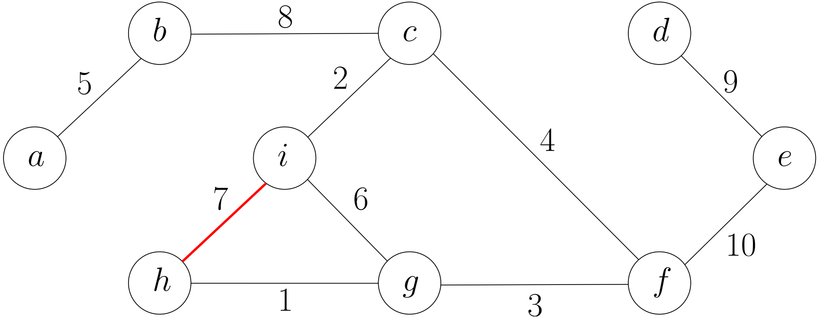
Reverse Kruskal's Algorithm: Example



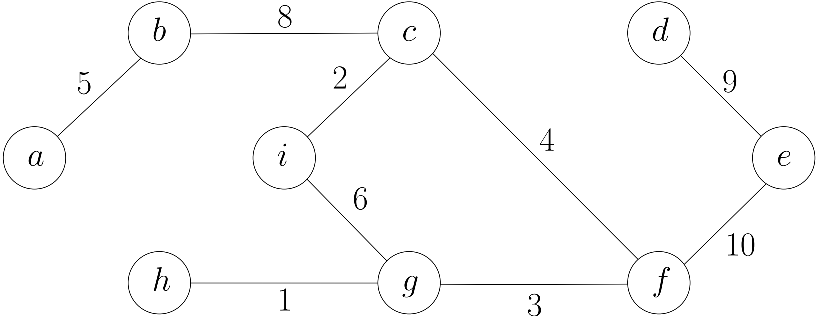
Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example



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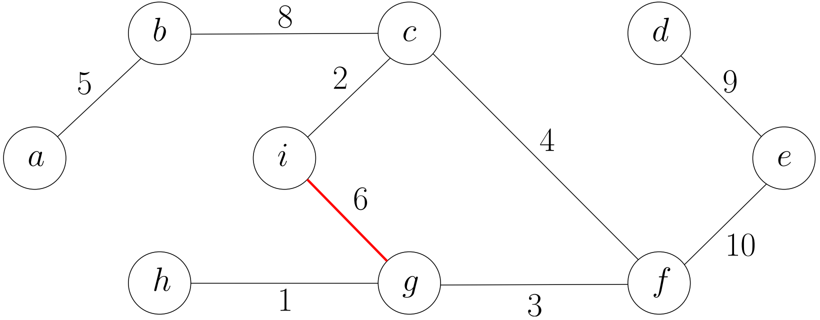

Reverse Kruskal's Algorithm: Example