Outline

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

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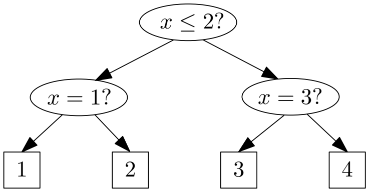

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

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- You can ask Bob "yes/no" questions about π.

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A: $\log_2 n! = \Theta(n \lg n)$

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- You can ask Bob questions of the form "does i appear before j in π?"

Q: How many questions do you need to ask in order to get the permutation π?

A: At least $\log_2 n! = \Theta(n \lg n)$

Outline

Selection Problem

Input: a set A of n numbers, and $1 \le i \le n$

Output: the i-th smallest number in A

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- Sorting solves the problem in time $O(n \lg n)$.
- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ ▷ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ ▷ Conquer
$t \leftarrow$ number of times x appear A
return the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Selection Algorithm with Median Finder

selection(A, n, i)

if $n = 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \le A_L$.size **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
else
    return x

Selection Algorithm with Median Finder

selection(A, n, i)

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A_L ← elements in A that are less than x                          ▷ Divide
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- Recurrence for selection: $T(n) = T(n/2) + O(n)$

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

if $n = 1$ **thenreturn** A
$x \leftarrow$ random element of A (called pivot)
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L.\text{size}$ **then**
    return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
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- expected running time $= O(n)$

Outline

Polynomial Multiplication

Input: two polynomials of degree $n-1$

Output: product of two polynomials

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Polynomial Multiplication

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Output: product of two polynomials

Example:

$$\begin{aligned}
&(3x^3+2x^2-5x+4)\times(2x^3-3x^2+6x-5)\\
=\ &6x^6-9x^5+18x^4-15x^3\\
&+4x^5-6x^4+12x^3-10x^2\\
&-10x^4+15x^3-30x^2+25x\\
&+8x^3-12x^2+24x-20\\
=\ &6x^6-5x^5+2x^4+20x^3-52x^2+49x-20
\end{aligned}$$

Polynomial Multiplication

Input: two polynomials of degree $n-1$

Output: product of two polynomials

Example:

$$
\begin{aligned}
&(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\
= {} & 6x^6 - 9x^5 + 18x^4 - 15x^3 \\
& + 4x^5 - 6x^4 + 12x^3 - 10x^2 \\
& - 10x^4 + 15x^3 - 30x^2 + 25x \\
& + 8x^3 - 12x^2 + 24x - 20 \\
= {} & 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}
$$

- **Input**: $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output**: $(-20, 49, -52, 20, 2, -5, 6)$

Naïve Algorithm

polynomial-multiplication(A, B, n)

let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
        $C[i+j] \leftarrow C[i+j] + A[i] \times B[j]$
return C

Naïve Algorithm

polynomial-multiplication(A, B, n)

```
let C[k] ← 0 for every k = 0, 1, 2, ··· , 2n − 2
for i ← 0 to n − 1 do
    for j ← 0 to n − 1 do
        C[i + j] ← C[i + j] + A[i] × B[j]
return C
```

Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x+2)x^2 + (-5x+4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x-3)x^2 + (6x-5)$$

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- $p(x)$: degree of $n-1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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$$pq = \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right)$$

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- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right)x^{n/2} + p_L q_L \end{aligned}$$

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$$\begin{aligned} \mathsf{multiply}(p, q) = {} & \mathsf{multiply}(p_H, q_H) \times x^n \\ & + \big(\mathsf{multiply}(p_H, q_L) + \mathsf{multiply}(p_L, q_H)\big) \times x^{n/2} \\ & + \mathsf{multiply}(p_L, q_L) \end{aligned}$$

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- Recurrence: $T(n) = 4T(n/2) + O(n)$

Divide-and-Conquer for Polynomial Multiplication

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- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

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\end{aligned}
$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

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$$r_H = \mathsf{multiply}(p_H, q_H)$$
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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \, .. \, n/2 - 1], A_H \leftarrow A[n/2 \, .. \, n - 1]$
$B_L \leftarrow B[0 \, .. \, n/2 - 1], B_H \leftarrow B[n/2 \, .. \, n - 1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n - 1)$ 0's
for $i \leftarrow 0$ to $n - 2$ **do**
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i + n] \leftarrow C[i + n] + C_H[i]$
    $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
return C