Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i) \in E} \{f(j) + w(j, i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

Shortest Paths in DAG

```
f[1] ← 0
for i ← 2 to n do
    f[i] ← ∞
    for each incoming edge (j, i) of i do
        if f[j] + w(j, i) < f[i] then
            f[i] ← f[j] + w(j, i)
```

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

Shortest Paths in DAG

$f[1] \leftarrow 0$
for $i \leftarrow 2$ to n **do**
    $f[i] \leftarrow \infty$
    for each incoming edge (j, i) of i **do**
        if $f[j] + w(j, i) < f[i]$ **then**
            $f[i] \leftarrow f[j] + w(j, i)$
            $\pi(i) \leftarrow j$

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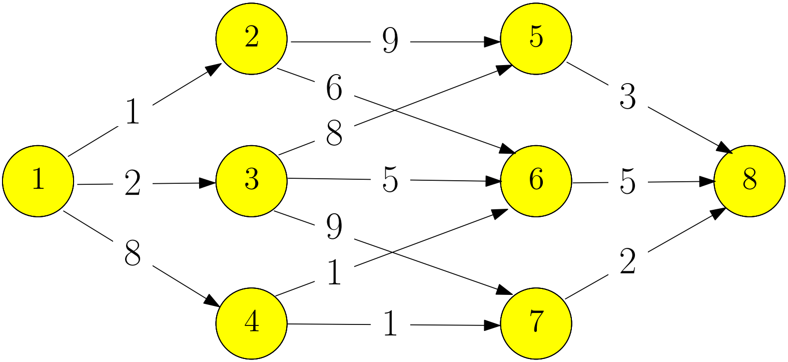
        if $f[j] + w(j, i) < f[i]$ **then**
            $f[i] \leftarrow f[j] + w(j, i)$
            $\pi(i) \leftarrow j$

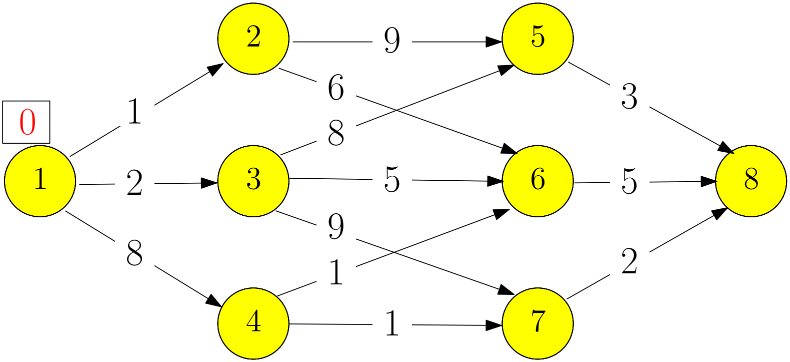
print-path(t)

if $t = 1$ **then**
    print(1)
    return
print-path($\pi(t)$)
print(",", t)

Example



Example



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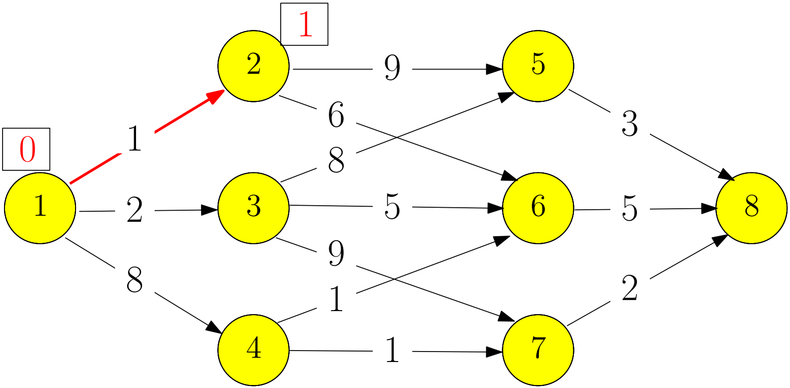

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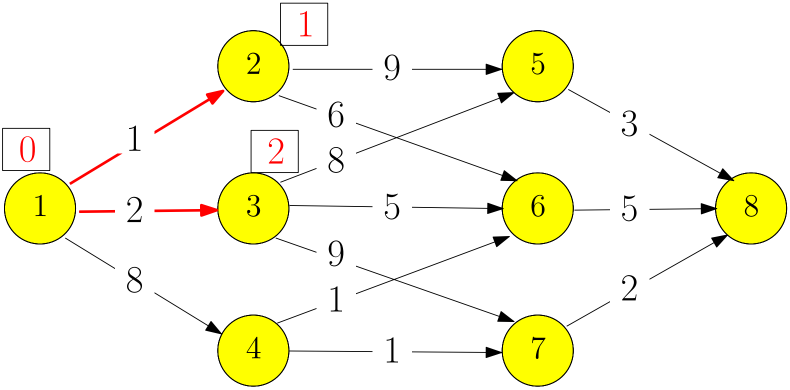

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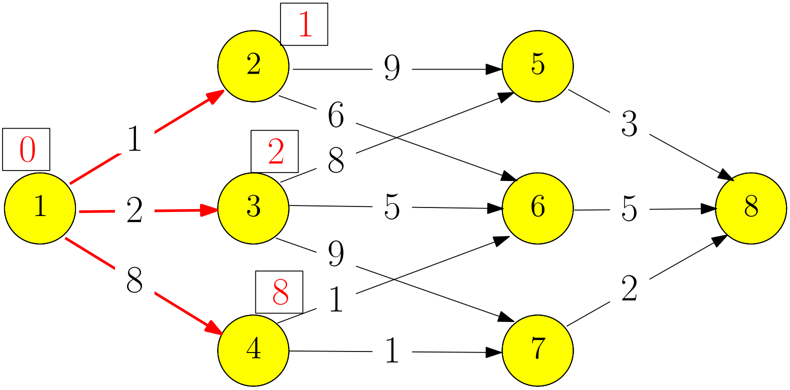

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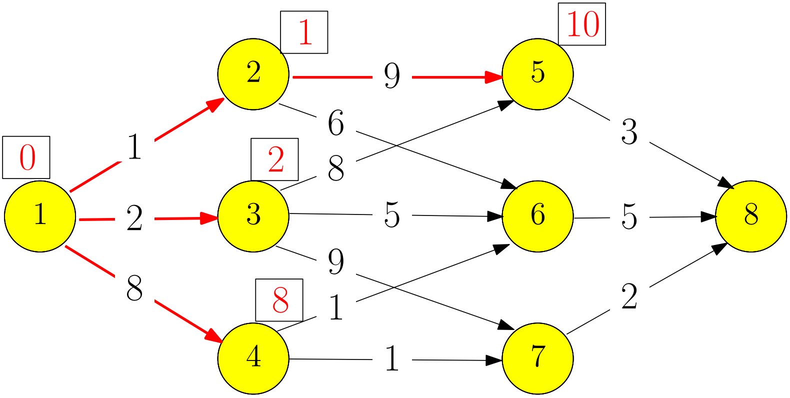

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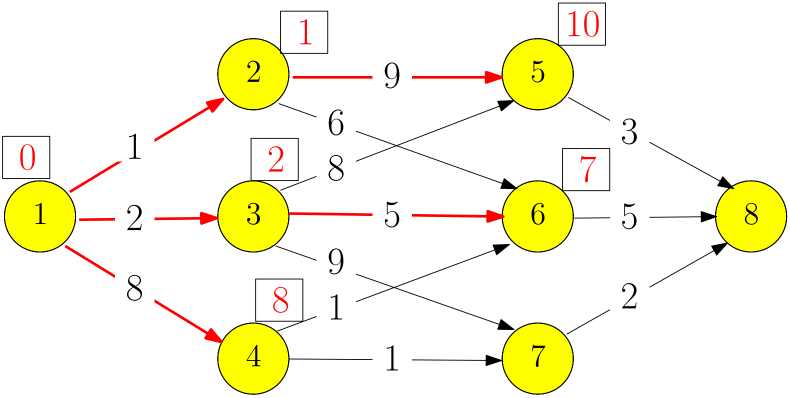

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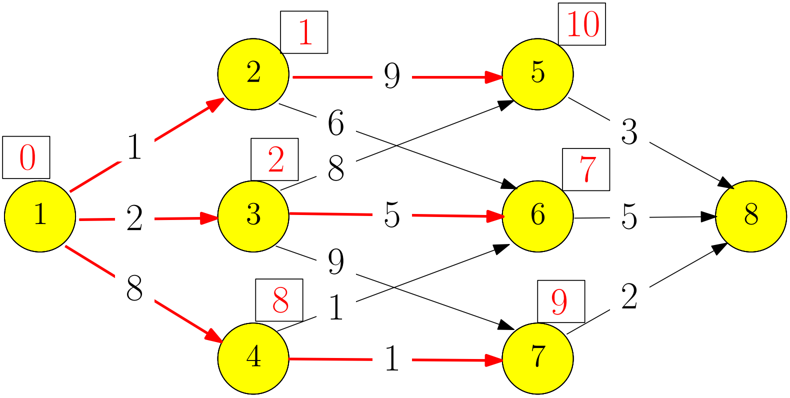

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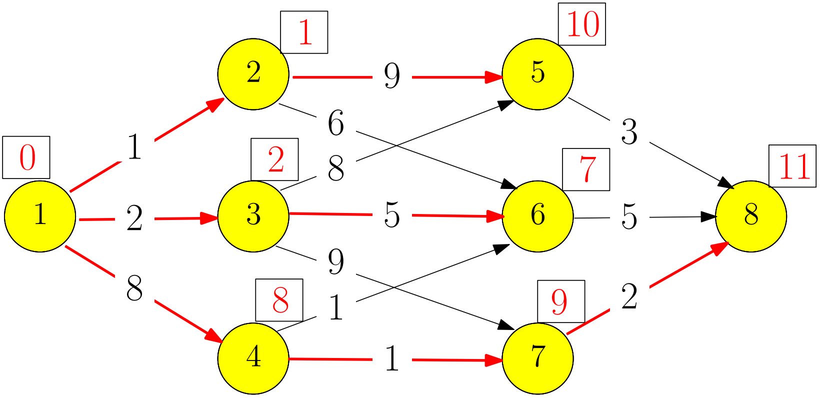

Variant: Heaviest Path in a Directed Acyclic Graph

Heaviest Path in a Directed Acyclic Graph

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the path with the largest weight (the heaviest path) from 1 to n.

- $f[i]$: weight of the heaviest path from 1 to i

$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

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$$f[i] = \begin{cases} 0 & i = 1 \\ \max_{j:(j,i)\in E} \{f(j) + w(j, i)\} & i = 2, 3, \cdots, n \end{cases}$$

Outline

Matrix Chain Multiplication

Matrix Chain Multiplication

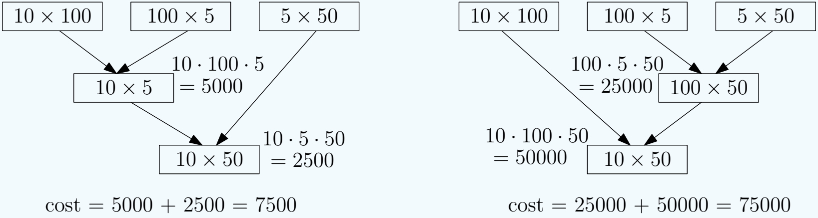
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$



- $(A_1A_2)A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1(A_2A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

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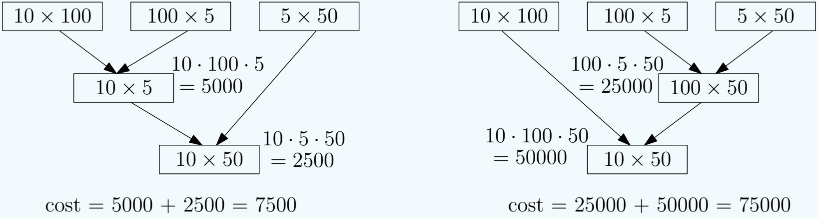

- $(A_1 A_2) A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1 (A_2 A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Matrix Chain Multiplication: Design DP

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- Assume the last step is $(A_1 A_2 \cdots A_i)(A_{i+1} A_{i+2} \cdots A_n)$

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- Optimality for sub-instances: we need to compute $A_1 A_2 \cdots A_i$ and $A_{i+1} A_{i+2} \cdots A_n$ optimally
- $opt[i, j]$: the minimum cost of computing $A_i A_{i+1} \cdots A_j$

$$opt[i, j] = \begin{cases} 0 & i = j \\ \min_{k: i \le k < j} \left(opt[i, k] + opt[k+1, j] + r_i c_k c_j \right) & i < j \end{cases}$$

Matrix Chain Multiplication: Design DP

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

let $opt[i,i] \leftarrow 0$ for every $i = 1, 2, \cdots, n$
for $\ell \leftarrow 2$ to n **do**
    for $i \leftarrow 1$ to $n - \ell + 1$ **do**
        $j \leftarrow i + \ell - 1$
        $opt[i,j] \leftarrow \infty$
        for $k \leftarrow i$ to $j - 1$ **do**
            if $opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j]$ **then**
                $opt[i,j] \leftarrow opt[i,k] + opt[k+1,j] + r_i c_k c_j$
return $opt[1,n]$

Recover the Optimum Way of Multiplication

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

```
let opt[i,i] ← 0 for every i = 1, 2, ..., n
for ℓ ← 2 to n do
    for i ← 1 to n − ℓ + 1 do
        j ← i + ℓ − 1
        opt[i,j] ← ∞
        for k ← i to j − 1 do
            if opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j] then
                opt[i,j] ← opt[i,k] + opt[k+1,j] + r_i c_k c_j
                π[i,j] ← k
return opt[1,n]
```

Constructing Optimal Solution

Print-Optimal-Order(i, j)

if $i = j$ **then**
    print("A"$_i$)
else
    print("(")
    Print-Optimal-Order$(i, \pi[i, j])$
    Print-Optimal-Order$(\pi[i, j] + 1, j)$
    print(")")

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[1,2] &= opt[1,1] + opt[2,2] + 3 \times 5 \times 2 = 30, && \pi[1,2] = 1 \\
opt[2,3] &= opt[2,2] + opt[3,3] + 5 \times 2 \times 6 = 60, && \pi[2,3] = 2 \\
opt[3,4] &= opt[3,3] + opt[4,4] + 2 \times 6 \times 9 = 108, && \pi[3,4] = 3 \\
opt[4,5] &= opt[4,4] + opt[5,5] + 6 \times 9 \times 4 = 216, && \pi[4,5] = 4 \\
opt[1,3] &= \min\{opt[1,1] + opt[2,3] + 3 \times 5 \times 6, \\
&\qquad\quad opt[1,2] + opt[3,3] + 3 \times 2 \times 6\} \\
&= \min\{0 + 60 + 90, 30 + 0 + 36\} = 66, && \pi[1,3] = 2 \\
opt[2,4] &= \min\{opt[2,2] + opt[3,4] + 5 \times 2 \times 9, \\
&\qquad\quad opt[2,3] + opt[4,4] + 5 \times 6 \times 9\} \\
&= \min\{0 + 108 + 90, 60 + 0 + 270\} = 198, && \pi[2,4] = 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[3,5] &= \min\{opt[3,3] + opt[4,5] + 2 \times 6 \times 4, \\
&\qquad opt[3,4] + opt[5,5] + 2 \times 9 \times 4\} \\
&= \min\{0 + 216 + 48, 108 + 0 + 72\} = 180, \\
\pi[3,5] &= 4, \\
opt[1,4] &= \min\{opt[1,1] + opt[2,4] + 3 \times 5 \times 9, \\
&\qquad opt[1,2] + opt[3,4] + 3 \times 2 \times 9, \\
&\qquad opt[1,3] + opt[4,4] + 3 \times 6 \times 9\} \\
&= \min\{0 + 198 + 135, 30 + 108 + 54, 66 + 0 + 162\} = 192, \\
\pi[1,4] &= 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[2,5] &= \min\{opt[2,2] + opt[3,5] + 5 \times 2 \times 4, \\
&\qquad opt[2,3] + opt[4,5] + 5 \times 6 \times 4, \\
&\qquad opt[2,4] + opt[5,5] + 5 \times 9 \times 4\} \\
&= \min\{0 + 180 + 40, 60 + 216 + 120, 198 + 0 + 180\} = 220, \\
opt[1,5] &= \min\{opt[1,1] + opt[2,5] + 3 \times 5 \times 4, \\
&\qquad opt[1,2] + opt[3,5] + 3 \times 2 \times 4, \\
&\qquad opt[1,3] + opt[4,5] + 3 \times 6 \times 4, \\
&\qquad opt[1,4] + opt[5,5] + 3 \times 9 \times 4\} \\
&= \min\{0 + 220 + 60, 30 + 180 + 24, \\
&\qquad 66 + 216 + 72, 192 + 0 + 108\} \\
&= 234, \\
\pi[1,5] &= 2.
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

opt, π	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	0, /	30, 1	66, 2	192, 2	234, 2
$i=2$		0, /	60, 2	198, 2	220, 2
$i=3$			0, /	108, 3	180, 4
$i=4$				0, /	216, 4
$i=5$					0, /

Print-Optimal-Order(1,5)
    Print-Optimal-Order(1, 2)
        Print-Optimal-Order(1, 1)
        Print-Optimal-Order(2, 2)
    Print-Optimal-Order(3, 5)
        Print-Optimal-Order(3, 4)
            Print-Optimal-Order(3, 3)
            Print-Optimal-Order(4, 4)
        Print-Optimal-Order(5, 5)

Optimum way for multiplication: $((A_1 A_2)((A_3 A_4) A_5))$

Outline

Optimum Binary Search Tree

Def. Binary search tree (BST), also called an ordered or sorted binary tree, is a rooted binary tree data structure with the key of each internal node being greater than all the keys in the respective node's left subtree and less than the ones in its right subtree.

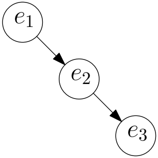
Optimum Binary Search Tree

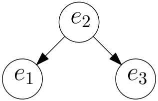
- n elements $e_1 < e_2 < e_3 < \cdots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \cdots, e_n\}$ with the minimum accessing cost:

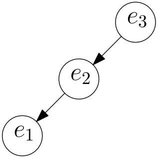
$$\sum_{i=1}^{n} f_i \times (\text{depth of } e_i \text{ in the tree})$$

Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$







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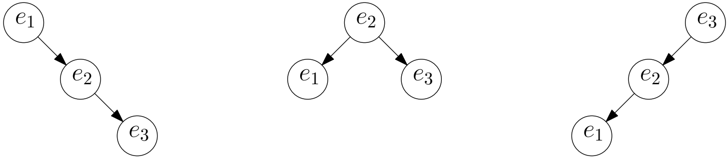

- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

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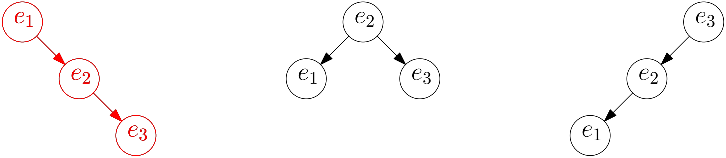

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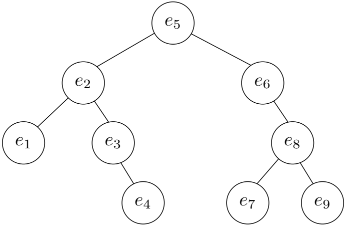
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

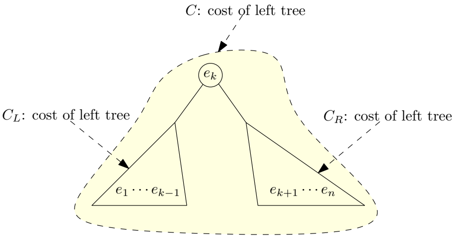
- suppose we decided to let e_k be the root
- $e_1, e_2, \cdots, e_{k-1}$ are on left sub-tree
- $e_{k+1}, e_{k+2}, \cdots, e_n$ are on right sub-tree

- d_j: depth of e_j in our tree
- C, C_L, C_R: cost of tree, left sub-tree and right sub-tree



- $d_1 = 3, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 1,$
- $d_6 = 2, d_7 = 4, d_8 = 3, d_9 = 4,$
- $C = 3f_1 + 2f_2 + 3f_3 + 4f_4 + f_5 + 2f_6 + 4f_7 + 3f_8 + 4f_9$
- $C_L = 2f_1 + f_2 + 2f_3 + 3f_4$
- $C_R = f_6 + 3f_7 + 2f_8 + 3f_9$

- $C = C_L + C_R + \sum_{j=1}^{9} f_j$



$$
\begin{aligned}
C = \sum_{\ell=1}^{n} f_\ell d_\ell &= \sum_{\ell=1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= \sum_{\ell=1}^{k-1} f_\ell (d_\ell - 1) + \sum_{\ell=k+1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= C_L + C_R + \sum_{\ell=1}^{n} f_\ell
\end{aligned}
$$

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- In order to minimize C, need to minimize C_L and C_R respectively

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$$opt[1, n] = \qquad (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

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$$opt[1, n] = \min_{k:1 \le k \le n} (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

- In general, $opt[i, j] =$

$$\begin{cases} 0 & \text{if } i = j+1 \\ \min_{k:i \le k \le j} \left(opt[i, k-1] + opt[k+1, j]\right) + \sum_{\ell=i}^{j} f_\ell & \text{if } i \le j \end{cases}$$

Optimum Binary Search Tree

$fsum[0] \leftarrow 0$

for $i \leftarrow 1$ to n **do** $fsum[i] \leftarrow fsum[i-1] + f_i$ $\quad \triangleright\ fsum[i] = \sum_{j=1}^{i} f_j$

for $i \leftarrow 0$ to n **do** $opt[i+1, i] \leftarrow 0$

for $\ell \leftarrow 1$ to n **do**

  for $i \leftarrow 1$ to $n - \ell + 1$ **do**

    $j \leftarrow i + \ell - 1$, $opt[i, j] \leftarrow \infty$

    for $k \leftarrow i$ to j **do**

      if $opt[i, k-1] + opt[k+1, j] < opt[i, j]$ **then**

        $opt[i, j] \leftarrow opt[i, k-1] + opt[k+1, j]$

        $\pi[i, j] \leftarrow k$

    $opt[i, j] \leftarrow opt[i, j] + fsum[j] - fsum[i-1]$

Printing the Tree

Print-Tree(i, j)

```
if i > j then
    return
else
    print('(')
    Print-Tree(i, π[i, j] − 1)
    print(π[i, j])
    Print-Tree(π[i, j] + 1, j)
    print(')')
```