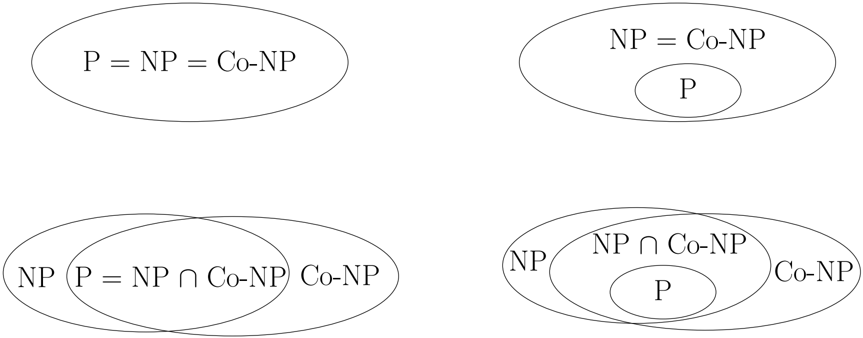
4 Possibilities of Relationships

Notice that $X \in \text{NP} \iff \overline{X} \in \text{Co-NP}$ and $\text{P} \subseteq \text{NP} \cap \text{Co-NP}$



- People commonly believe we are in the 4th scenario

Outline

Polynomial-Time Reductions

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \le_P X$.

Polynomial-Time Reductions

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

To prove positive results:

Suppose $Y \leq_P X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.

Polynomial-Time Reductions

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

To prove positive results:

Suppose $Y \leq_P X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.

To prove negative results:

Suppose $Y \leq_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

Output: whether there is a Hamiltonian path from s to t in G

Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

Output: whether there is a Hamiltonian path from s to t in G

Lemma HP $\leq_{\mathsf{P}}$ HC.

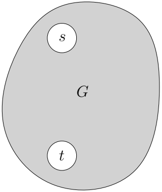
Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

Output: whether there is a Hamiltonian path from s to t in G

Lemma HP $\leq_{\mathsf{P}}$ HC.



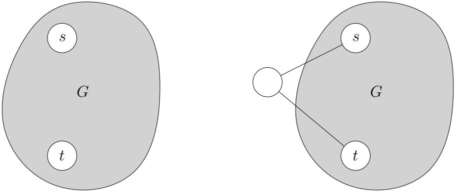
Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

Output: whether there is a Hamiltonian path from s to t in G

Lemma HP $\leq_{\mathsf{P}}$ HC.



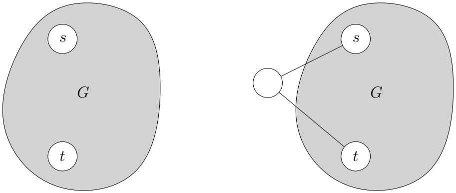
Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

Output: whether there is a Hamiltonian path from s to t in G

Lemma HP $\leq_{\text{P}}$ HC.



Obs. G has a HP from s to t if and only if graph on right side has a HC.

NP-Completeness

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_\text{P} X$ for every $Y \in \text{NP}$.

NP-Completeness

Def. A problem X is called NP-hard if

2. $Y \leq_{\mathrm{P}} X$ for every $Y \in \mathrm{NP}$.

- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)

NP-Completeness

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)

NP-Completeness

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

Theorem If X is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)

NP-Completeness

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

Theorem If X is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP
- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)

Outline

Def. A problem X is called NP-complete if

1. $X \in \mathrm{NP}$, and
2. $Y \leq_{\mathrm{P}} X$ for every $Y \in \mathrm{NP}$.

Def. A problem X is called NP-complete if

1. $X \in \mathrm{NP}$, and
2. $Y \leq_{\mathrm{P}} X$ for every $Y \in \mathrm{NP}$.

- How can we find a problem $X \in \mathrm{NP}$ such that every problem $Y \in \mathrm{NP}$ is polynomial time reducible to X? Are we asking for too much?

Def. A problem X is called NP-complete if

1. $X \in \mathrm{NP}$, and
2. $Y \leq_{\mathrm{P}} X$ for every $Y \in \mathrm{NP}$.

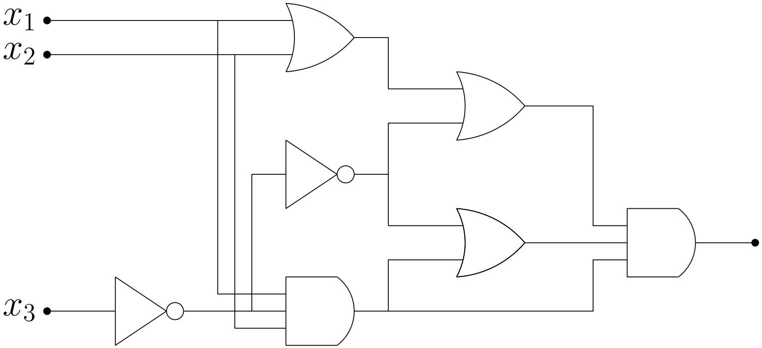
- How can we find a problem $X \in \mathrm{NP}$ such that every problem $Y \in \mathrm{NP}$ is polynomial time reducible to X? Are we asking for too much?
- No! There is indeed a large family of natural NP-complete problems

The First NP-Complete Problem: Circuit-Sat

Circuit Satisfiability (Circuit-Sat)

Input: a circuit

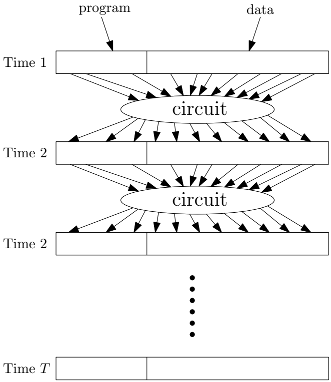
Output: whether the circuit is satisfiable



Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

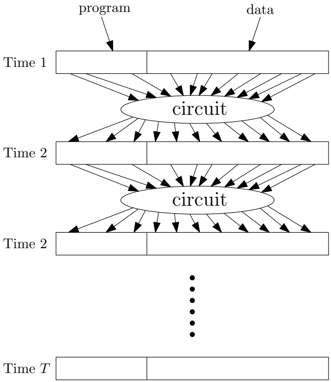
Fact Any algorithm that takes n bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

Fact Any algorithm that takes n bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



- Then, we can show that any problem $Y \in$ NP can be reduced to Circuit-Sat.
- We prove HC $\leq_P$ Circuit-Sat as an example.

HC $\leq_P$ Circuit-Sat

check-HC(G, S)

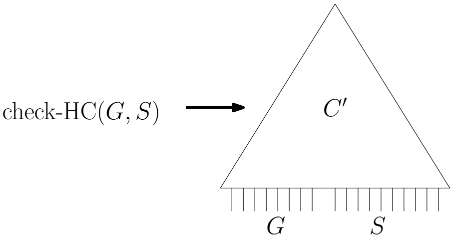
- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.

HC $\leq_P$ Circuit-Sat

check-HC(G, S)

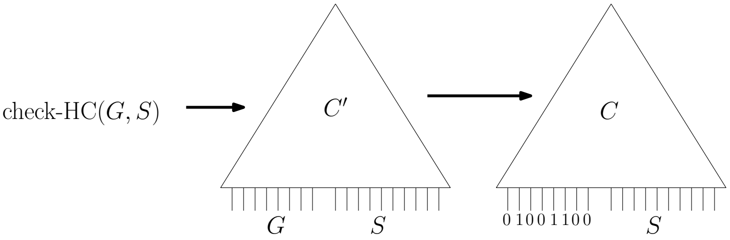
- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.
- G is a yes-instance if and only if there is an S such that check-HC(G, S) returns 1

HC $\leq_P$ Circuit-Sat



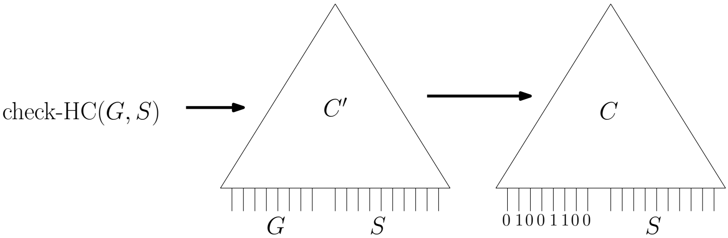
- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.
- G is a yes-instance if and only if there is an S such that check-HC(G, S) returns 1
- Construct a circuit C' for the algorithm check-HC

HC $\leq_P$ Circuit-Sat



- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.
- G is a yes-instance if and only if there is an S such that check-HC(G, S) returns 1
- Construct a circuit C' for the algorithm check-HC
- hard-wire the instance G to the circuit C' to obtain the circuit C

HC $\leq_P$ Circuit-Sat



- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.
- G is a yes-instance if and only if there is an S such that check-HC(G, S) returns 1
- Construct a circuit C' for the algorithm check-HC
- hard-wire the instance G to the circuit C' to obtain the circuit C
- G is a yes-instance if and only if C is satisfiable

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

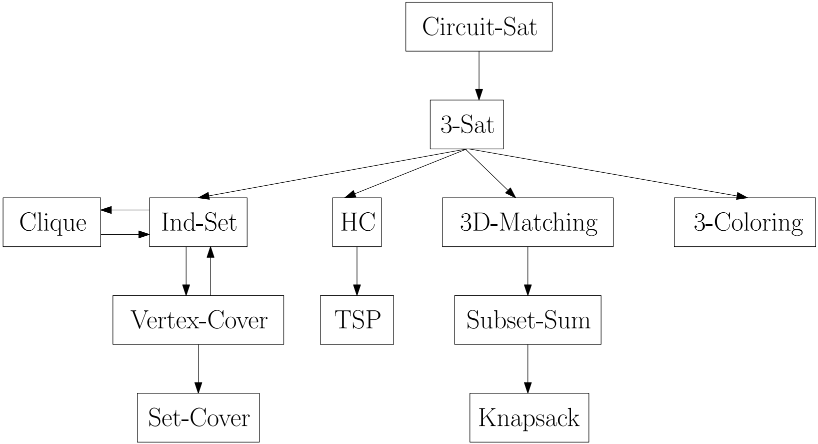
- Let check-Y(s,t) be the certifier for problem Y: check-Y(s,t) returns 1 if t is a valid certificate for s.
- s is a yes-instance if and only if there is a t such that check-Y(s,t) returns 1
- Construct a circuit C' for the algorithm check-Y
- hard-wire the instance s to the circuit C' to obtain the circuit C
- s is a yes-instance if and only if C is satisfiable $\square$

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

- Let check-Y(s,t) be the certifier for problem Y: check-Y(s,t) returns 1 if t is a valid certificate for s.
- s is a yes-instance if and only if there is a t such that check-Y(s,t) returns 1
- Construct a circuit C' for the algorithm check-Y
- hard-wire the instance s to the circuit C' to obtain the circuit C
- s is a yes-instance if and only if C is satisfiable $\square$

Theorem Circuit-Sat is NP-complete.

Reductions of NP-Complete Problems



3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$

3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: x_i or $\neg x_i$

3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: x_i or $\neg x_i$
- Clause: disjunction ("or") of at most 3 literals: $x_3 \vee \neg x_4$, $x_1 \vee x_8 \vee \neg x_9$, $\neg x_2 \vee \neg x_5 \vee x_7$

3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: x_i or $\neg x_i$
- Clause: disjunction ("or") of at most 3 literals: $x_3 \vee \neg x_4$, $x_1 \vee x_8 \vee \neg x_9, \quad \neg x_2 \vee \neg x_5 \vee x_7$
- 3-CNF formula: conjunction ("and") of clauses: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

3-Sat

3-Sat

Input: a 3-CNF formula

Output: whether the 3-CNF is satisfiable

3-Sat

3-Sat

Input: a 3-CNF formula

Output: whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses

3-Sat

3-Sat

Input: a 3-CNF formula

Output: whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal

3-Sat

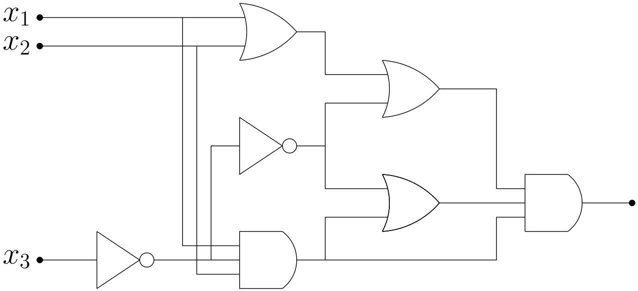
3-Sat

Input: a 3-CNF formula

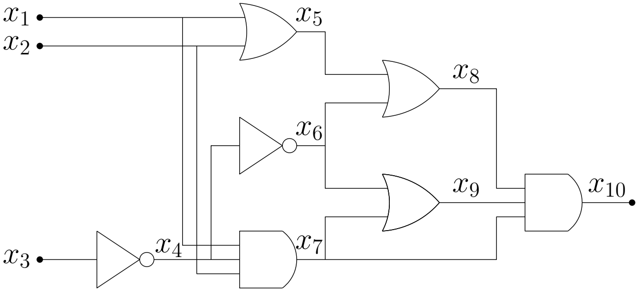
Output: whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal
- Assignment $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ satisfies $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

Circuit-Sat $\leq_P$ 3-Sat

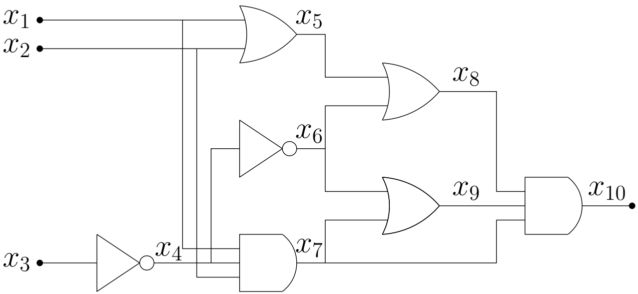


Circuit-Sat $\leq_P$ 3-Sat



- Associate every wire with a new variable

Circuit-Sat $\leq_P$ 3-Sat



- Associate every wire with a new variable
- The circuit is equivalent to the following formula:

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Circuit-Sat $\leq_P$ 3-Sat

$$
\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}
$$

Convert each clause to a 3-CNF

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$
\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}
$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_7) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee \neg x_2 \vee x_5)$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF

Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable

Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit

Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit
- Thus, Circuit-Sat $\leq_P$ 3-Sat

Reductions of NP-Complete Problems