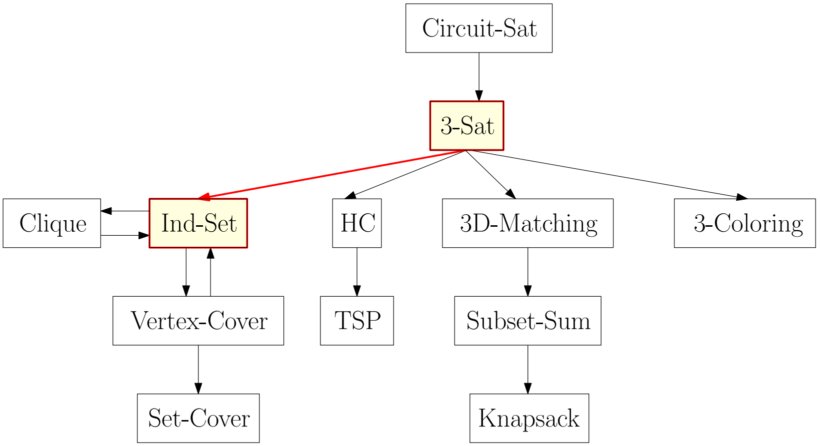
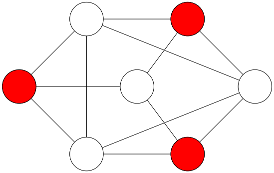
Reductions of NP-Complete Problems



Recall: Independent Set Problem

Def. An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G.



Independent Set (Ind-Set) Problem

Input: $G = (V, E), k$

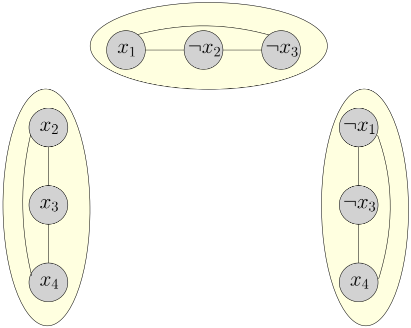
Output: whether there is an independent set of size k in G

3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$
- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group



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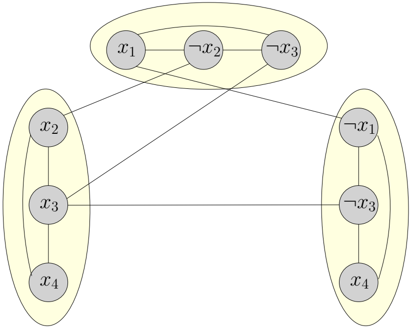

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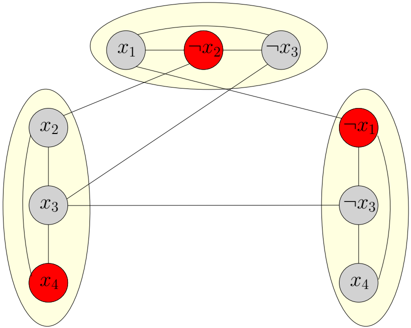

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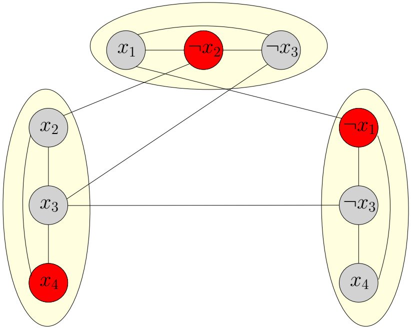

3-Sat instance is yes-instance $\Leftrightarrow$ Ind-Set instance is yes-instance:

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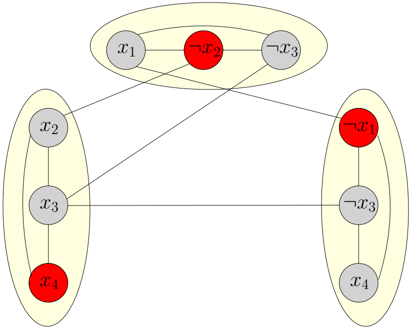
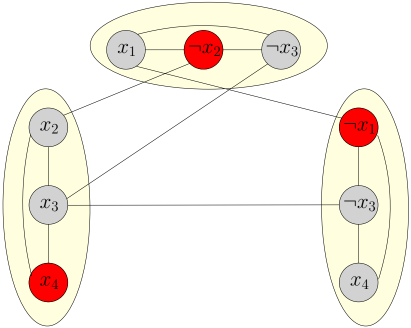


3-Sat instance is yes-instance $\Leftrightarrow$ Ind-Set instance is yes-instance:

- satisfying assignment $\Rightarrow$ independent set of size k
- independent set of size k $\Rightarrow$ satisfying assignment

Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



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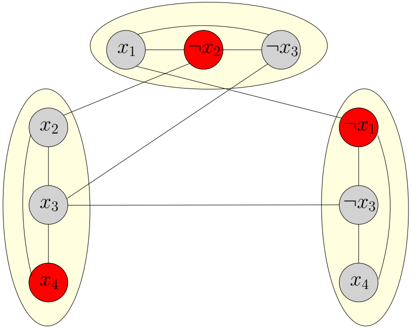
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$
- For every clause, at least 1 literal is satisfied



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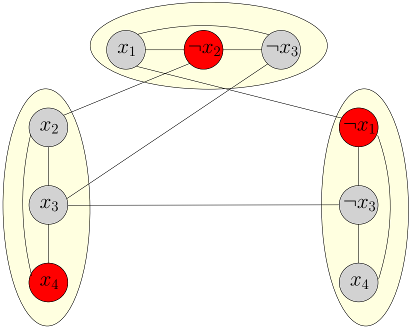
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal



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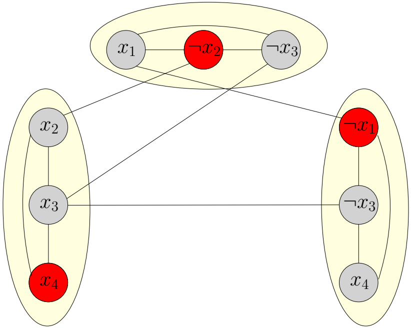
- So, 1 literal from each group



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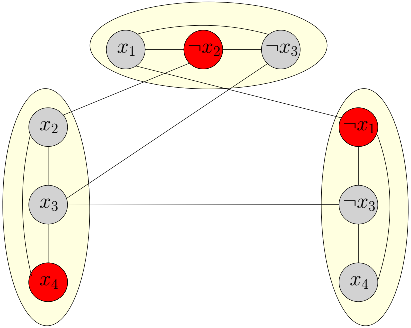
- So, 1 literal from each group
- No contradictions among the selected literals



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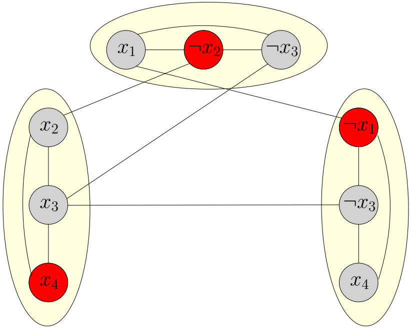
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals
- An IS of size k



IS of Size $k \Rightarrow$ Satisfying Assignment

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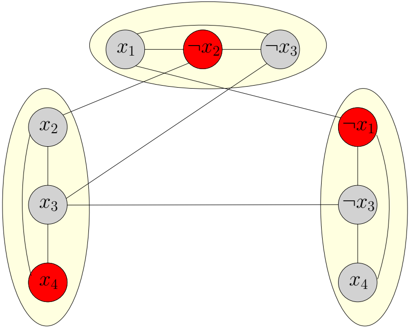
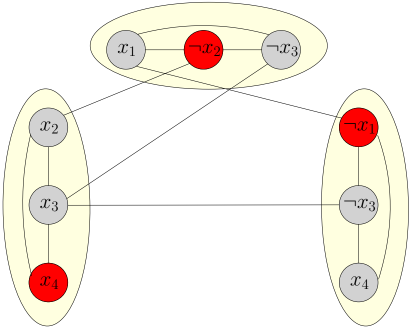


IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every group, exactly one literal is selected in IS



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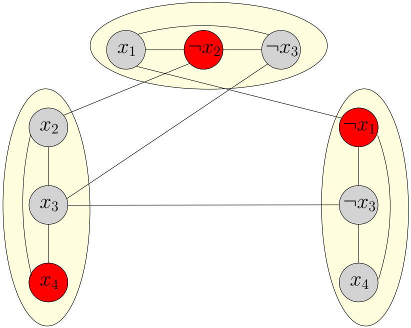

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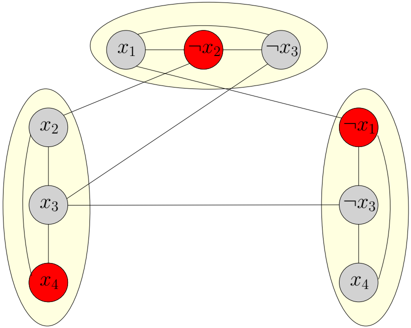
- If x_i is selected in IS, set $x_i = 1$



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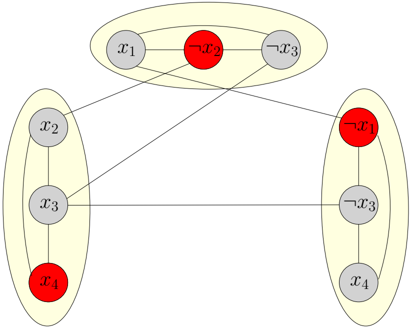

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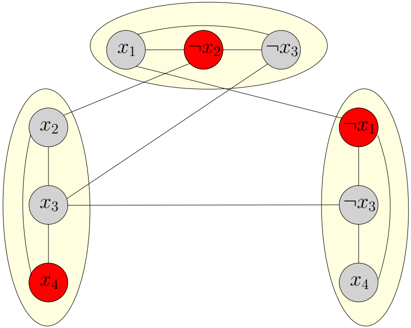
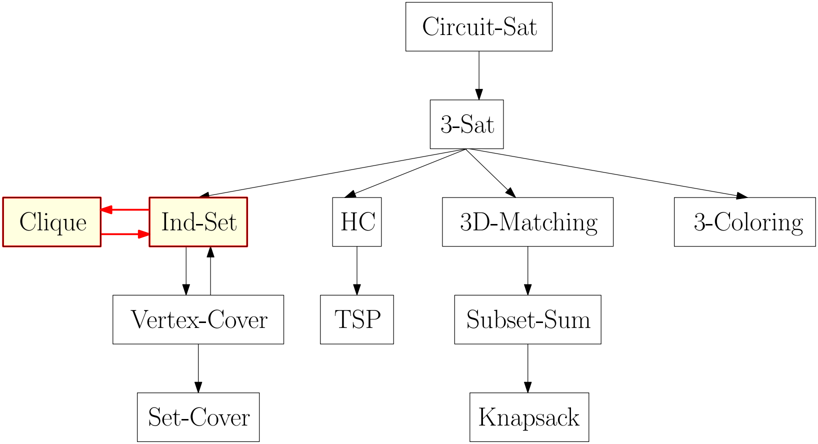
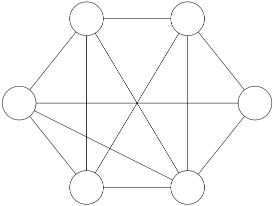
- Otherwise, set x_i arbitrarily



Reductions of NP-Complete Problems



Def. A clique in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$



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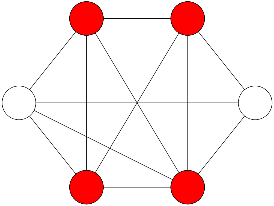

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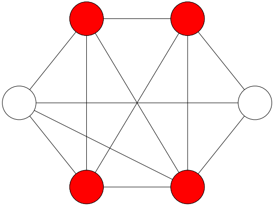

Clique Problem

Input: $G = (V, E)$ and integer $k > 0$,

Output: whether there exists a clique of size k in G

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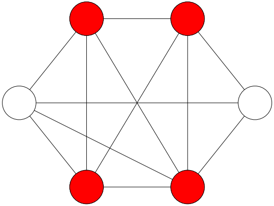

Clique Problem

Input: $G = (V, E)$ and integer $k > 0$,

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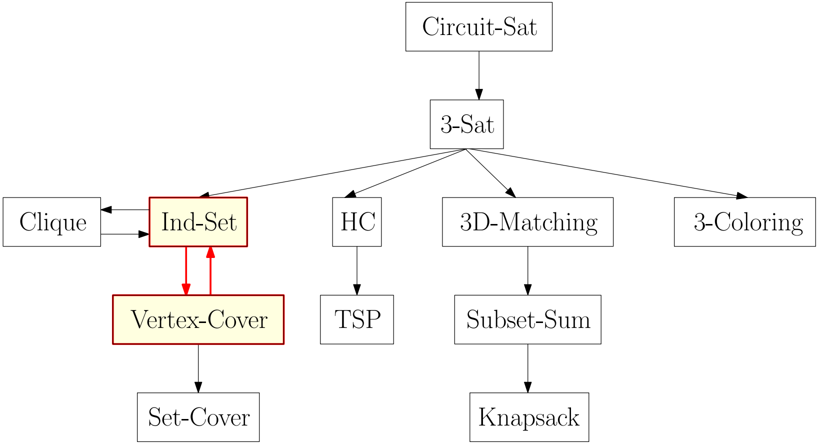
- What is the relationship between Clique and Ind-Set?

Clique $=_P$ Ind-Set

Def. Given a graph $G = (V, E)$, define $\overline{G} = (V, \overline{E})$ be the graph such that $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.

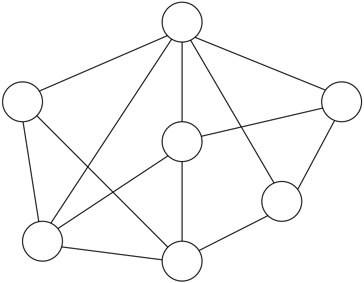
Obs. S is an independent set in G if and only if S is a clique in $\overline{G}$.

Reductions of NP-Complete Problems



Vertex-Cover

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



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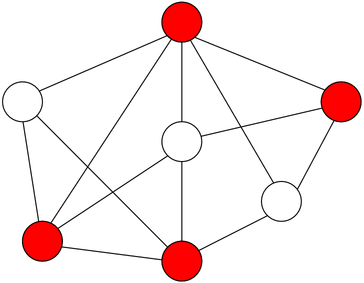

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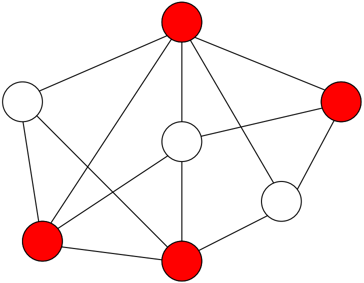

Vertex-Cover Problem

Input: $G = (V, E)$ and integer k

Output: whether there is a vertex cover of G of size at most k

Vertex-Cover $=_P$ Ind-Set

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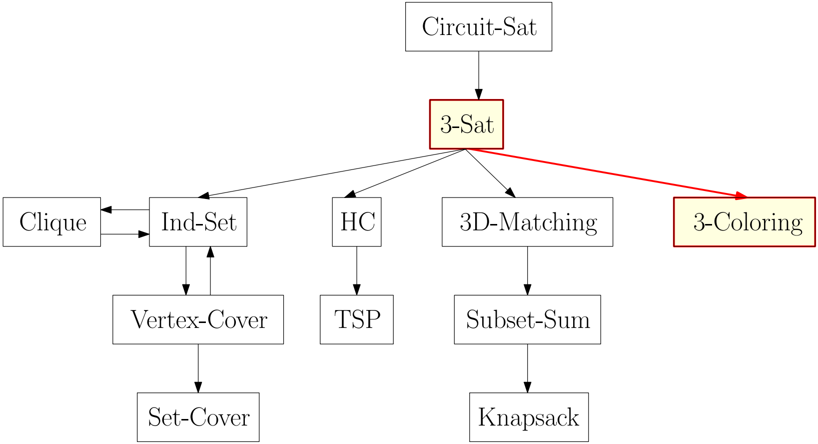
Q: What is the relationship between Vertex-Cover and Ind-Set?

Vertex-Cover $=_P$ Ind-Set

Q: What is the relationship between Vertex-Cover and Ind-Set?

A: S is a vertex-cover of $G = (V, E)$ if and only if $V \setminus S$ is an independent set of G.

Reductions of NP-Complete Problems



A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

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- In general, algorithm for Y can call the algorithm for X many times.
- However, for most reductions, we call algorithm for X only once
- That is, for a given instance s_Y for Y, we only construct one instance s_X for X

A Strategy of Polynomial Reduction

- Given an instance s_Y of problem Y, show how to construct in polynomial time an instance s_X of problem such that:
  - s_Y is a yes-instance of $Y \Rightarrow s_X$ is a yes-instance of X
  - s_X is a yes-instance of $X \Rightarrow s_Y$ is a yes-instance of Y

Outline

Q: How far away are we from proving or disproving $\mathrm{P} = \mathrm{NP}$?

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- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.

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  - Assume the number of clauses is $\Theta(n)$, $n =$ number variables

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  - Best lower bound is $\Omega(n)$
- Essentially we have no techniques for proving lower bound for running time

Dealing with NP-Hard Problems

- Faster exponential time algorithms
- Solving the problem for special cases
- Fixed parameter tractability
- Approximation algorithms

Faster Exponential Time Algorithms

3-SAT:

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- Brute-force: $O(2^n \cdot \mathsf{poly}(n))$

Faster Exponential Time Algorithms

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Travelling Salesman Problem:

Faster Exponential Time Algorithms

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Travelling Salesman Problem:

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Faster Exponential Time Algorithms

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Travelling Salesman Problem:

- Brute-force: $O(n! \cdot \text{poly}(n))$
- Better algorithm: $O(2^n \cdot \text{poly}(n))$
- In practice: TSP Solver can solve Euclidean TSP instances with more than 100,000 vertices

Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on

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Collaborative delivery problem (reduction from 3DM) is NP-hard on general graphs, but easy on

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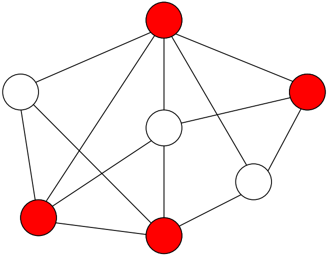
Fixed Parameter Tractability

- Problem: whether there is a vertex cover of size k, for a small k (number of nodes is n, number of edges is $\Theta(n)$.)



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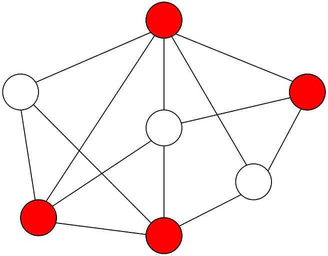

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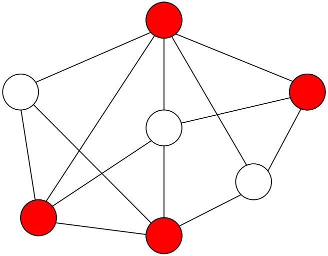

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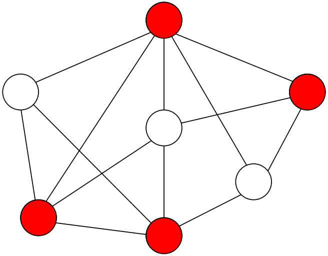

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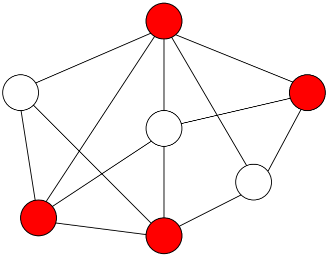
- Vertex-Cover is fixed-parameter tractable.



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- There is an 2-approximation for the vertex cover problem: we can efficiently find a vertex cover whose size is at most 2 times that of the optimal vertex cover

2-Approximation Algorithm for Vertex Cover

VertexCover(G)

1: $C \leftarrow \emptyset$
2: **while** $\neq \emptyset$ **do**
3: select an edge $(u, v) \in E$, $C \leftarrow C \cup \{u, v\}$
4: Remove from E every edge incident on either u or v
5: **return** C

- Let the set C and C^* be the sets output by above algorithm and an optimal alg, respectively. Let S be the set of edges selected.
- Since no two edge in S are covered by the same vertex (Once an edge is picked in line 3, all other edges that are incident on its endpoints are removed from E in line 4), we have $|C^*| \geq |S|$;
- As we have added both vertices of edge (u, v), we get $|C| = 2|S|$ but C^* have to add one of the two, thus, $|C|/|C^*| \leq 2$.