Outline

1. Syllabus

2. Introduction
    - What is an Algorithm?
    - Example: Insertion Sort
    - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times

Sorting Problem

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

Insertion-Sort

- At the end of j-th iteration, the first j numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 53 & 59 & 15 \\ & & & & \uparrow & \\ & & & & i & \end{array}$$

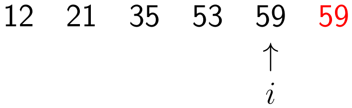
Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$



12 21 35 53 59 59

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 53 & 59 & 59 \\ & & & \uparrow & & \\ & & & i & & \end{array}$$

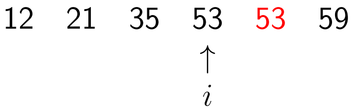
Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$


12 21 35 53 53 59

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$

12 21 35 53 53 59

$\uparrow$

i

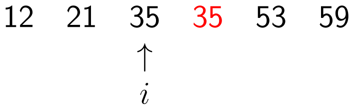
Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$



Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$

12 21 35 35 53 59

↑ i (under 21)

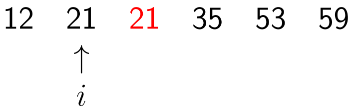
Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$



Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$

12 21 21 35 53 59

↑

i

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- $j = 6$
- $key = 15$

12 15 21 35 53 59

↑

i

Outline

1. Syllabus

2. Introduction
    - What is an Algorithm?
    - Example: Insertion Sort
    - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times

Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 5 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph

- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph
- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
- A2: Worst-case analysis:
  - Running time for size $n =$ worst running time over all possible arrays of length n

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?
- A: They do not matter!

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?
- A: They do not matter!

Important idea: asymptotic analysis

- Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$
- $n^2/100 - 3n + 10 = O(n^2)$

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or # instructions?

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or $\#$ instructions?
- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or 10^{-8} seconds
  - program 2 requires 2 instructions, or 10^{-9} seconds

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or $\#$ instructions?
- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or 10^{-8} seconds
  - program 2 requires 2 instructions, or 10^{-9} seconds
  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.

- For Algorithm 1: if we increase n by a factor of 2, running time increases by a factor of 4

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.

- For Algorithm 1: if we increase n by a factor of 2, running time increases by a factor of 4
- For Algorithm 2: if we increase n by a factor of 2, running time increases by a factor of 2

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

Asymptotic Analysis of Insertion Sort

```
insertion-sort$(A, n)$
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- Worst-case running time for iteration j of the outer loop?

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

```
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- Worst-case running time for iteration j of the outer loop?
  Answer: $O(j)$

Asymptotic Analysis of Insertion Sort

```
insertion-sort$(A, n)$
for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$
```

- Worst-case running time for iteration j of the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)$
  $= O(\frac{n(n+1)}{2} - 1) = O(n^2)$

Computation Model

Computation Model

- Random-Access Machine (RAM) model
    - reading and writing $A[j]$ takes $O(1)$ time

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$?

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$? No

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50?$ No
- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

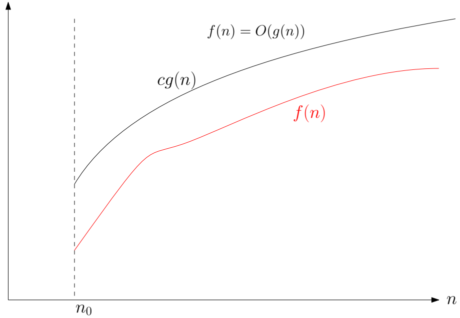
- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c > 0$ and every large enough n.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c > 0$ and every large enough n.



O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c > 0$ and every large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c > 0$ and every large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$

Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$
$$= -n^2 + 42n \leq 0.$$
$$3n^2 + 2n \leq c(n^2 - 10n)$$

□

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- $n^{100} \in O(2^n)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- $n^{100} \in O(2^n)$
- $n^3 \notin O(10n^2)$

O-Notation For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- $n^{100} \in O(2^n)$
- $n^3 \notin O(10n^2)$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\le$		

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$

"$=$" is asymmetric! Following equalities are wrong:

- $O(n^3 - 10n) = 3n^2 + 2n$
- $O(n^2 + 5n) = 3n^2 + 2n$
- $O(n^2) = 3n^2 + 2n$

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$

"$=$" is asymmetric! Following equalities are wrong:

- $O(n^3 - 10n) = 3n^2 + 2n$
- $O(n^2 + 5n) = 3n^2 + 2n$
- $O(n^2) = 3n^2 + 2n$

- Analogy: Mike is a student. ~~A student is Mike.~~

Ω-Notation: Asymptotic Lower Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\big\}.$$

Ω-**Notation** For a function $g(n)$,

$$\Omega(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \ge cg(n), \forall n \ge n_0\big\}.$$

Ω-Notation: Asymptotic Lower Bound

O-Notation For a function $g(n)$,

$$O(g(n)) = \Big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\Big\}.$$

Ω-Notation For a function $g(n)$,

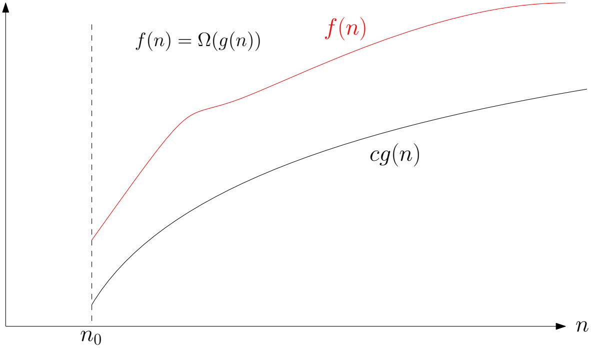
$$\Omega(g(n)) = \Big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \ge cg(n), \forall n \ge n_0\Big\}.$$

- In other words, $f(n) \in \Omega(g(n))$ if $f(n) \ge cg(n)$ for some c and large enough n.

Ω-Notation: Asymptotic Lower Bound

Ω-Notation For a function $g(n)$,

$$\Omega(g(n)) = \Big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\Big\}.$$



Ω-Notation: Asymptotic Lower Bound

- Again, we use "$=$" instead of $\in$.
  - $4n^2 = \Omega(n - 10)$
  - $3n^2 - n + 10 = \Omega(n^2 - 20)$

Ω-Notation: Asymptotic Lower Bound

- Again, we use "$=$" instead of $\in$.
  - $4n^2 = \Omega(n - 10)$
  - $3n^2 - n + 10 = \Omega(n^2 - 20)$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$	$\geq$	

Ω-Notation: Asymptotic Lower Bound

- Again, we use "$=$" instead of $\in$.
  - $4n^2 = \Omega(n - 10)$
  - $3n^2 - n + 10 = \Omega(n^2 - 20)$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$	$\geq$	

Theorem $f(n) = O(g(n)) \ \Leftrightarrow \ g(n) = \Omega(f(n))$.

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \big\{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\big\}.$$

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \big\{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } \\ c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\big\}.$$

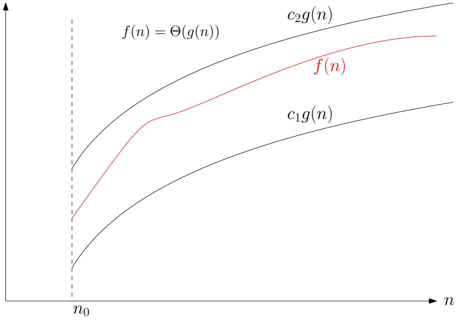
- $f(n) = \Theta(g(n))$, then for large enough n, we have "$f(n) \approx g(n)$".

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\}.$$

- $f(n) = \Theta(g(n))$, then for large enough n, we have "$f(n) \approx g(n)$".



Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \Big\{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\Big\}.$$

- $3n^2 + 2n = \Theta(n^2 - 20n)$

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \big\{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\big\}.$$

- $3n^2 + 2n = \Theta(n^2 - 20n)$
- $2^{n/3+100} = \Theta(2^{n/3})$

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \Big\{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\Big\}.$$

- $3n^2 + 2n = \Theta(n^2 - 20n)$
- $2^{n/3+100} = \Theta(2^{n/3})$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$	$\geq$	$=$

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\}.$$

- $3n^2 + 2n = \Theta(n^2 - 20n)$
- $2^{n/3+100} = \Theta(2^{n/3})$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$	$\geq$	$=$

Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.