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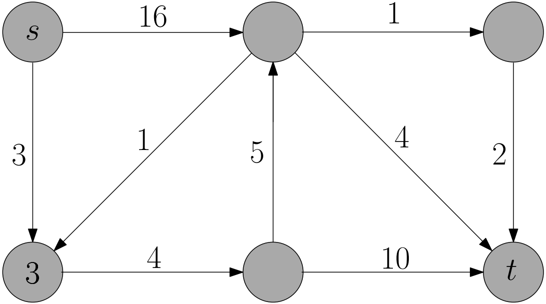

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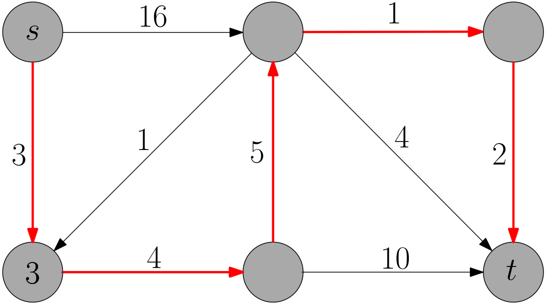

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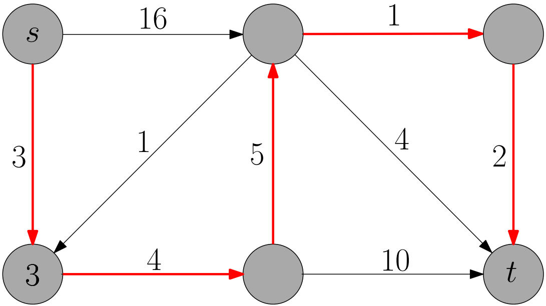

- Algorithm: Dijkstra's algorithm . . .

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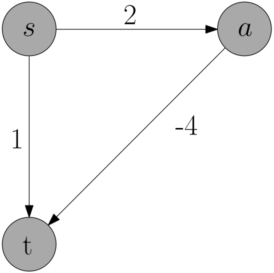

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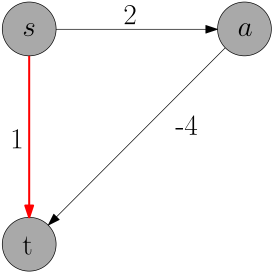

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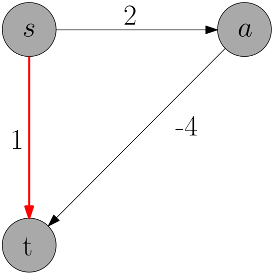

- Algorithm: Bellman-Ford algorithm, Floyd-Warshall . . .

# Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, using a particular programming language

# Pseudo-Code

Pseudo-Code:

**Euclidean$(a, b)$**

```
while $b > 0$ do
    $(a, b) \leftarrow (b, a \bmod b)$
return $a$
```

Python program:

```
def euclidean(a: int, b: int):
    c = 0
    while b > 0:
        c = b
        b = a % b
        a = c
    return a
```

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4. it is fun!

# Outline

1. Syllabus
2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

## Sorting Problem

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

**Output:** a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

## Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

# Insertion-Sort

- At the end of $j$-th iteration, the first $j$ numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

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- $j = 6$
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| | | | | | |
|---|---|---|---|---|---|
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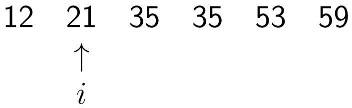

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# Analysis of Insertion Sort

- Correctness
- Running time

# Correctness of Insertion Sort

- Invariant: after iteration $j$ of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

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- A2: Worst-case analysis:
  - Running time for size $n$ = worst running time over all possible arrays of length $n$

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**Important idea: asymptotic analysis**

- Focus on growth of running-time as a function, not any particular value.

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  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

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- Can we do better than insertion sort asymptotically?

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  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer $n$ and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

# Outline

1. Syllabus
2. Introduction
    - What is an Algorithm?
    - Example: Insertion Sort
    - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

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- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
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- We only consider asymptotically positive functions.

# $O$-Notation: Asymptotic Upper Bound

$O$-**Notation** For a function $g(n)$,

$$O(g(n)) = \Big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\Big\}.$$

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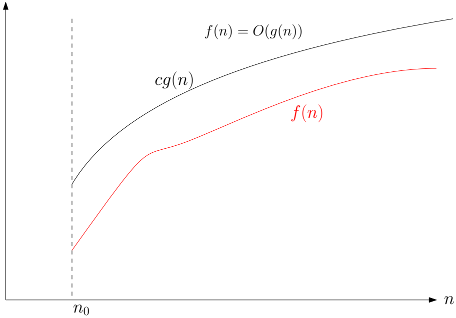

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**Proof.**

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$

$$= -n^2 + 42n \leq 0.$$

$$3n^2 + 2n \leq c(n^2 - 10n)$$

□

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| Asymptotic Notations | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|
| Comparison Relations | $\leq$ | | |