CSE 431/531: Algorithm Analysis and Design (Spring 2024)
Greedy Algorithms

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Def. In an optimization problem, our goal is to find a valid solution with the minimum cost (or maximum value).

Trivial Algorithm for an Optimization Problem
Enumerate all valid solutions, compare them and output the best one.

However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.

$f(n)$ is a polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.

convention: polynomial time = efficient

Goals of algorithm design
1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems
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Common Paradigms for Algorithm Design

- Greedy Algorithms: shortest path problem
- Divide and Conquer: merge-sort, binary search
- Dynamic Programming: shortest path problem, Fibonacci number
Greedy algorithm properties

Greedy algorithms are often for optimization problems. They often run in polynomial time due to their simplicity: easy to come up with, easy to analyze running time. Hard to see correctness. Mostly, it is not correct. E.g. min $f(x)$
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Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy
Greedy Algorithm

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- At each step, make an irreversible decision using a “reasonable” strategy

Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is “safe”
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
Greedy Algorithm

- Build up the solutions in steps
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Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is “safe” **(key)**
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem **(usually easy)**
Greedy Algorithm

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- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem *(usually easy)*

**Def.** A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.
Outline

1. Toy Example: Box Packing

2. Interval Scheduling
   - Interval Partitioning

3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue

4. Data Compression and Huffman Code

5. Summary

6. Exercise Problems
Box Packing

**Input:** \( n \) boxes of capacities \( c_1, c_2, \cdots, c_n \)
\( m \) items of sizes \( s_1, s_2, \cdots, s_m \)

Can put at most 1 item in a box

Item \( j \) can be put into box \( i \) if \( s_j \leq c_i \)

**Output:** A way to put as many items as possible in the boxes.

**Example:**
- Box capacities: 60, 40, 25, 17, 12
- Item sizes: 45, 41, 20, 19, 16
Box Packing

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**Example:**
- Box capacities: 60, 40, 25, 17, 12
- Item sizes: 45, 41, 20, 19, 16
- Can put 3 items in boxes: 45 → 60, 20 → 40, 16 → 25
Box Packing

**Input:** $n$ boxes of capacities $c_1, c_2, \cdots, c_n$

$m$ items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item $j$ can be put into box $i$ if $s_j \leq c_i$

**Output:** A way to put as many items as possible in the boxes.

Example:

- Box capacities: 60, 40, 25, 17, 12
- Item sizes: 45, 41, 20, 19, 16
- Can put 3 items in boxes: 45 $\rightarrow$ 60, 20 $\rightarrow$ 40, 16 $\rightarrow$ 25
- Can put 4 items in boxes: 45 $\rightarrow$ 60, 20 $\rightarrow$ 40, 19 $\rightarrow$ 25, 16 $\rightarrow$ 17
Greedy Algorithm

- Build up the solutions in steps
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- Build up the solutions in steps
- At each step, make an irreversable decision using a “reasonable” strategy

Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
**Greedy Algorithm**

- Build up the solutions in steps
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**Designing a Reasonable Strategy for Box Packing**

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.
Analysis of Greedy Algorithm

- **Safety**: Prove that the reasonable strategy is “safe”
- **Self-reduce**: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

**Lemma**
The strategy that put into box 1 the largest item it can hold is “safe”:

There is an optimum solution in which box 1 contains the largest item it can hold.

**Intuition**: putting the item gives us the easiest residual problem.

**Formal proof via exchanging argument**:
Analysis of Greedy Algorithm

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- **Intuition**: putting the item gives us the easiest residual problem.
- **Formal proof via exchanging argument**: 
Lemma  There is an optimum solution in which box 1 contains the largest item it can hold.

Proof. Let \( j \) = largest item that box 1 can hold. Take any optimum solution \( S \). If \( j \) is put into Box 1 in \( S \), done. Otherwise, assume this is what happens in \( S \): \( s_j \not\leq s_j \), and swap giving a solution \( S_0 \). In \( S_0 \), \( j \) is put into Box 1.
**Lemma**  There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**

- Let $j =$ largest item that box 1 can hold.
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**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

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- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$:

  ![Diagram of box 1 with items](image.png)

  where $j$ is placed in box 1, and swapping gives a solution $S^\prime$. In $S^\prime$, $j$ is put into Box 1.
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**
- Let $j$ = largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$:
  - $s_{j'} \leq s_j$, and swapping gives another solution $S'$
Lemma  There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let $j = \text{largest item that box 1 can hold}$.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$: 

  $S'$:

  \[
  \begin{array}{cccc}
  \text{box 1} & \text{item } j' & \text{item } j & \cdots \n  \end{array}
  \]

  $s_{j'} \leq s_j$, and swapping gives another solution $S'$

  $S'$ is also an optimum solution. In $S'$, $j$ is put into Box 1.  \qed
Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.
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Analysis of Greedy Algorithm

- **Safety:** Prove that the reasonable strategy is “safe”
- **Self-reduce:** Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- **Trivial:** we decided to put Item $j$ into Box 1, and the remaining instance is obtained by removing Item $j$ and Box 1.
Generic Greedy Algorithm

1. while the instance is non-trivial do
2. make the choice using the greedy strategy
3. reduce the instance

Greedy Algorithm for Box Packing

1. \( T \leftarrow \{1, 2, 3, \cdots, m\} \)
2. for \( i \leftarrow 1 \) to \( n \) do
3. if some item in \( T \) can be put into box \( i \) then
4. \( j \leftarrow \) the largest item in \( T \) that can be put into box \( i \)
5. print(“put item \( j \) in box \( i \)”)
6. \( T \leftarrow T \setminus \{j\} \)
Why “Safety” + “Self-reduce” $\implies$ Optimality?

- Let $\text{BP}(B,T)$ denote a box-packing instance.
- $\phi(1, 2, ..., m) \mapsto \{1, 2, ..., n, \text{NULL}\}$ denote packing strategy. e.g., $\phi(2) = 3$ means item 2 is put into box 3.
- $\text{val}(\phi) :=$ the number of items packed by $\phi$.
- $\phi_g$: the packing strategy obtained by greedy algorithm.

**Proof.**

- **Base case:** When $|B| = 1$ or $|T| = 1$.
- **Inductive case:** (Hypothesis) Assume Greedy alg solves $\text{BP}(B', T')$ optimally for $|B'| = n - 1$ and $|T'| = m - 1$. 


Why “Safety” + “Self-reduce” $\implies$ Optimality?

Proof.

1. (Induction) Wlog, let $\pi$ be the optimal solution matches our greedy sol on $\text{BP}(B, T)$, saying $\pi(j) = 1$.
2. By self-reduce: $\text{BP}(B \setminus \{1\}, T \setminus \{j\})$ is a smaller BP instance.
3. $\pi$ and $\phi_g$ onto $\text{BP}(B \setminus \{1\}, T \setminus \{j\})$, denoted as $\pi'$ and $\phi'_g$.
4. By Inductive hypothesis, $\phi'_g$ is the optimal sol for $\text{BP}(B \setminus \{1\}, T \setminus \{j\})$.
5. $\text{val}(\pi) \geq \text{val}(\phi_g) = 1 + \text{val}(\phi'_g) \geq 1 + \text{val}(\pi') = \text{val}(\pi)$. 

$q.e.d.$
Running time

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- With sorted item-sizes and box-capacities, running time is $O(\max\{n, m\})$. 
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Lemma  Generic algorithm is correct if and only if the greedy strategy is safe.
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Lemma  Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy strategy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.
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Def. A strategy is “safe” if there is always an optimum solution that is “consistent” with the decision made according to the strategy.
let $S$ be an arbitrary optimum solution.

if $S$ is consistent with the greedy choice, done.

otherwise, show that it can be modified to another optimum solution $S'$ that is consistent with the choice.
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The procedure is not a part of the algorithm.
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Interval Scheduling

**Input:** \( n \) jobs, job \( i \) with start time \( s_i \) and finish time \( f_i \)

\( i \) and \( j \) are compatible if \([s_i, f_i)\) and \([s_j, f_j)\) are disjoint

**Output:** A maximum-size subset of mutually compatible jobs
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Greedy Algorithm for Interval Scheduling

Which of the following strategies are safe?

No!
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size?
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Schedule the job with the earliest finish time? Yes!
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![Diagram showing intervals and job scheduling]

- [Diagram details: intervals from 0 to 9, colors representing scheduled and available intervals]
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- Take an arbitrary optimum solution $S$

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- What is the remaining task after we decided to schedule $j$?
- Is it another instance of interval scheduling problem?
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**Schedule**\((s, f, n)\)

1: \(A \leftarrow \{1, 2, \cdots , n\}, \ S \leftarrow \emptyset\)
2: \textbf{while} \(A \neq \emptyset\) \textbf{do}
3: \(j \leftarrow \text{arg min}_{j' \in A} f_{j'}\)
4: \(S \leftarrow S \cup \{j\}; \ A \leftarrow \{j' \in A : s_{j'} \geq f_j\}\)
5: \textbf{return} \(S\)
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4. \(S \leftarrow S \cup \{j\}; A \leftarrow \{j' \in A : s_{j'} \geq f_j\}\)
5. **return** \(S\)
Greedy Algorithm for Interval Scheduling

\[\text{Schedule}(s, f, n)\]

1: \[A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset\]
2: \textbf{while} \( A \neq \emptyset \) \textbf{do}
3: \( j \leftarrow \arg \min_{j' \in A} f_{j'} \)
4: \( S \leftarrow S \cup \{j\}; A \leftarrow \{j' \in A : s_{j'} \geq f_j\}\)
5: \textbf{return} \( S \)

Running time of algorithm?
Greedy Algorithm for Interval Scheduling

**Schedule***(s, f, n)***

1. \( A \leftarrow \{1, 2, \ldots, n\} \), \( S \leftarrow \emptyset \)
2. while \( A \neq \emptyset \) do
3. \( j \leftarrow \arg \min_{j' \in A} f_{j'} \)
4. \( S \leftarrow S \cup \{j\} \); \( A \leftarrow \{j' \in A : s_{j'} \geq f_{j}\} \)
5. return \( S \)

Running time of algorithm?

- Naive implementation: \( O(n^2) \) time
Greedy Algorithm for Interval Scheduling

**Schedule**($s, f, n$)

1. $A \leftarrow \{1, 2, \cdots, n\}$, $S \leftarrow \emptyset$
2. while $A \neq \emptyset$ do
3. \hspace{1em} $j \leftarrow \text{arg min}_{j' \in A} f_{j'}$
4. \hspace{1em} $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5. return $S$

Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time