Terminologies

When we talk about upper bound on running time:

- Logarithmic time: $O(\log n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
  - $O(n \log n) \subseteq O(n^{1.1})$. So, an $O(n \log n)$-time algorithm is also a polynomial time algorithm.
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.

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- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

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- e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

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- Sometimes yes
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- For "natural" algorithms, constants are not so big!

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A:

- Sometimes yes
- However, when n is big enough, $1000n < 0.1n^2$
- For "natural" algorithms, constants are not so big!
- So, for reasonably large n, algorithm with lower order running time beats algorithm with higher order running time.

CSE 431/531: Algorithm Analysis and Design (Spring 2024)
Graph Basics

Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo

Outline

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

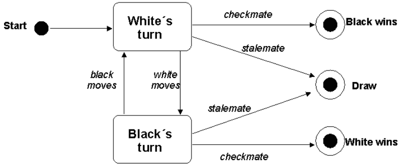
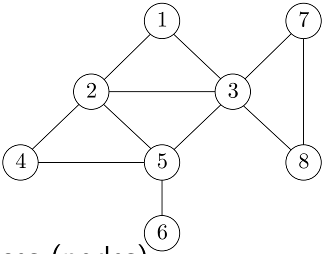


Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);

- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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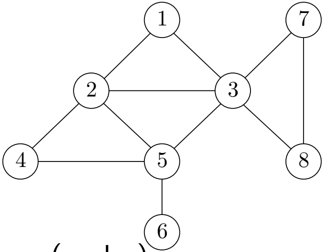
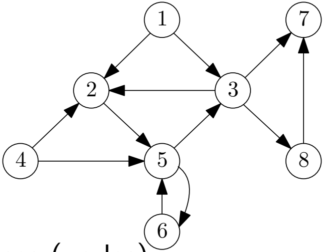


- V: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,5\}, \{3,7\}, \{3,8\}, \{4,5\}, \{5,6\}, \{7,8\}\}$

Directed Graph $G = (V, E)$



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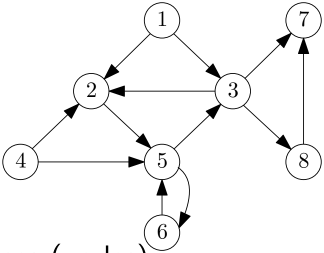

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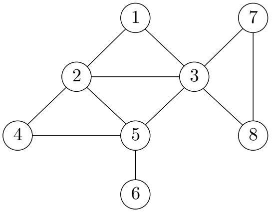
Abuse of Notations

- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



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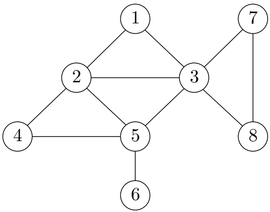
- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

Representation of Graphs



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
  - $n \times n$ matrix, $A[u,v] = 1$ if $(u,v) \in E$ and $A[u,v] = 0$ otherwise
  - A is symmetric if graph is undirected

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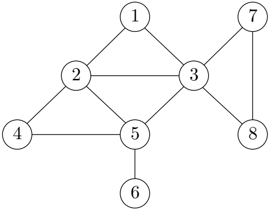

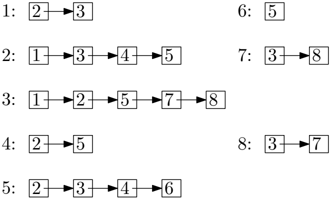

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  - For every vertex v, there is a linked list containing all neighbors of v.

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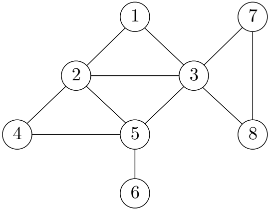

1: [2 | 3]
2: [1 | 3 | 4 | 5]
3: [1 | 2 | 5 | 7 | 8]
4: [2 | 5]
5: [2 | 3 | 4 | 6]
6: [5]
7: [3 | 8]
8: [3 | 7]

$d : (2, 4, 5, 2, 4, 1, 2, 2)$

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  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbors of v.
  - When graph is static, can use array of variant-length arrays.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
time to list all neighbors of v		

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memory usage	$O(n^2)$	$O(m)$
time to check $(u,v) \in E$	$O(1)$	
time to list all neighbors of v		

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Outline

Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

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Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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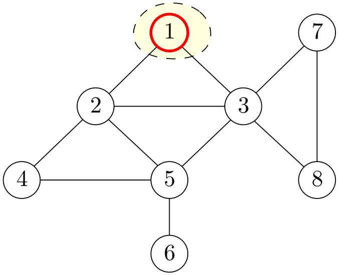

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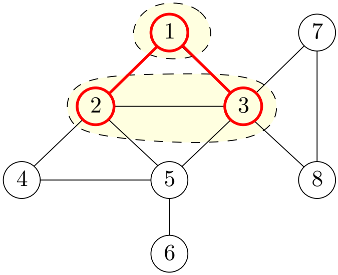

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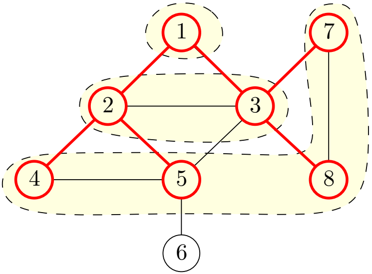

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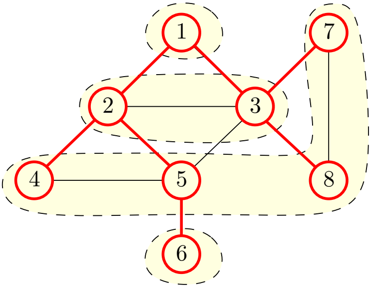


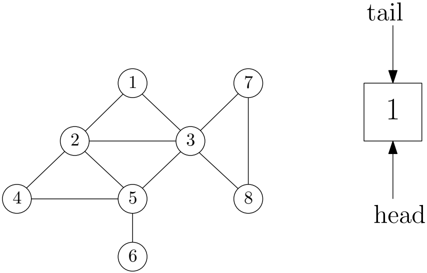
Implementing BFS using a Queue

BFS(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark s as "visited" and all other vertices as "unvisited"
while $head \leq tail$ **do**
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbors u of v **do**
        if u is "unvisited" **then**
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark u as "visited"

- Running time: $O(n + m)$.

Example of BFS via Queue



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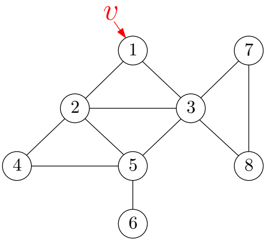

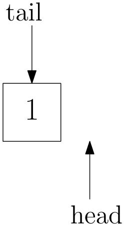

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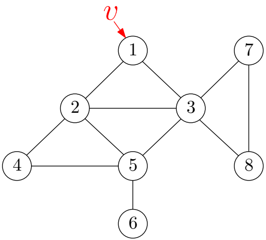

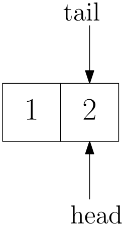

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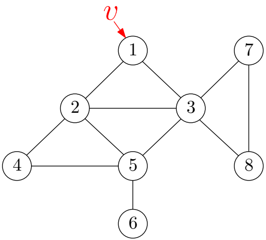

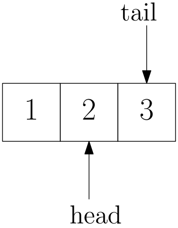

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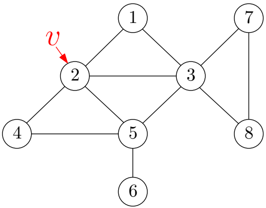

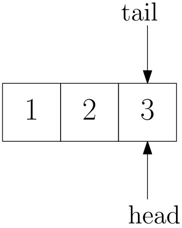

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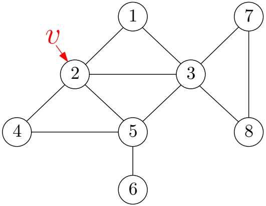

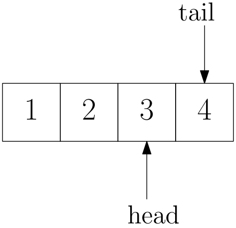

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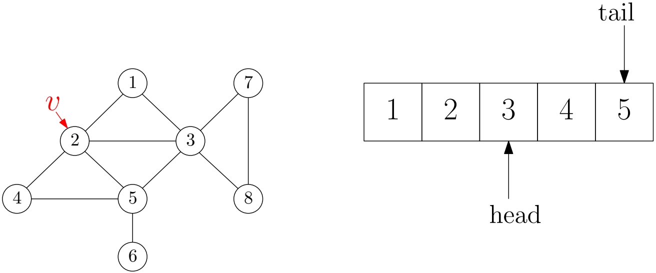

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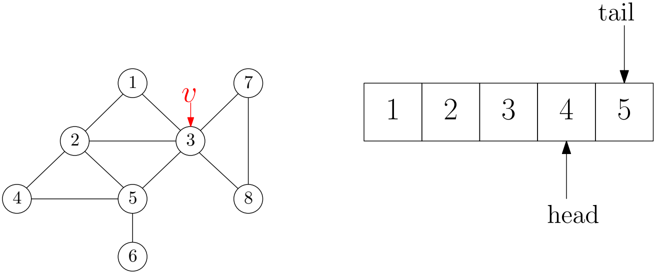

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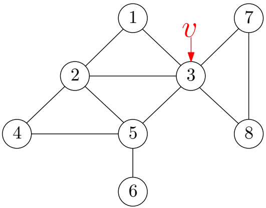

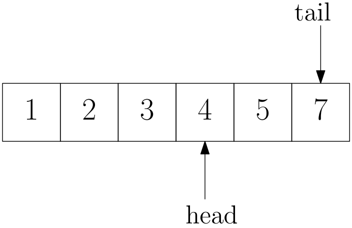

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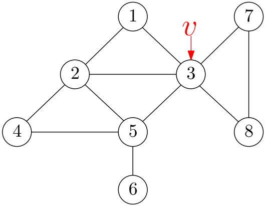

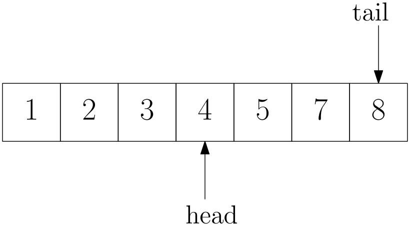

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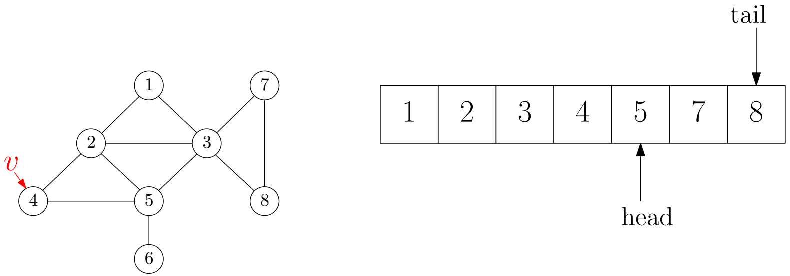

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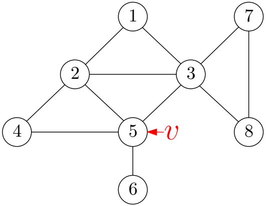

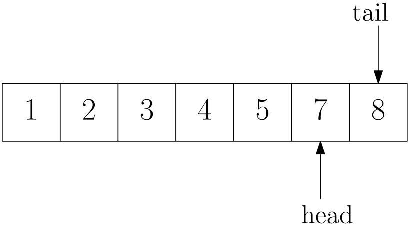

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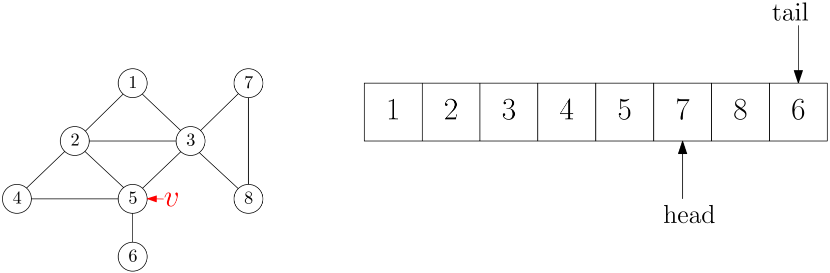

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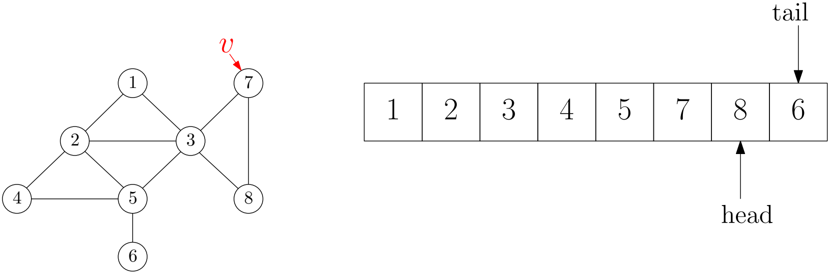

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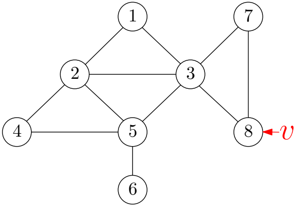

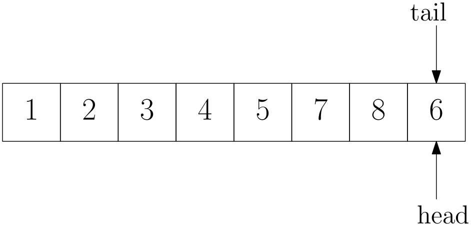

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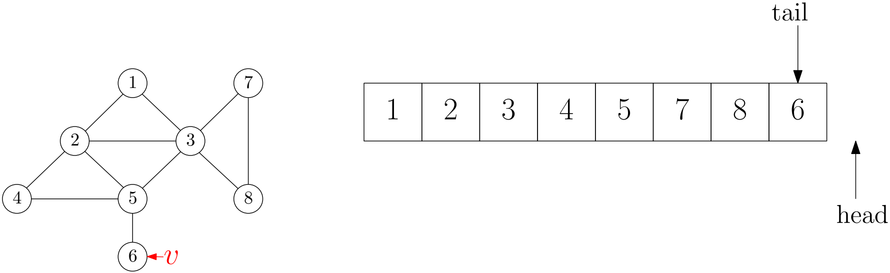

Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex ("dead-end"), go back
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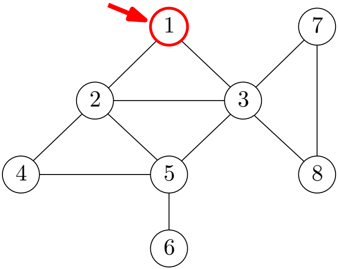

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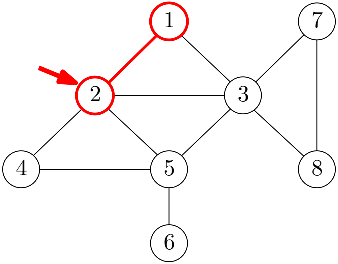

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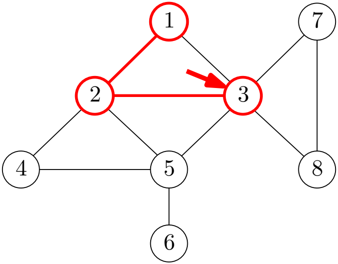

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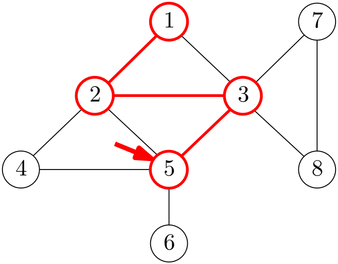

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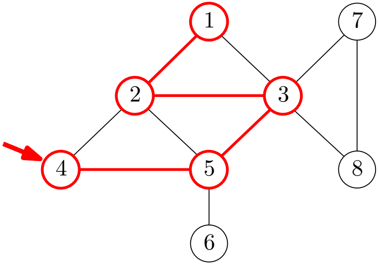

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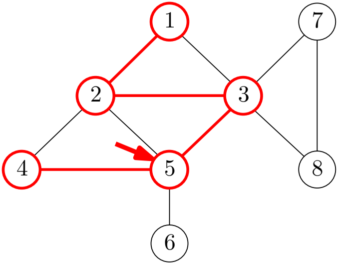

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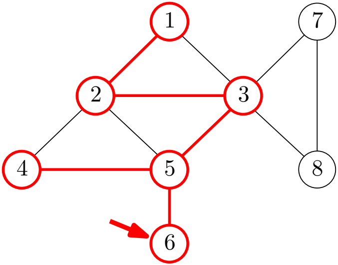

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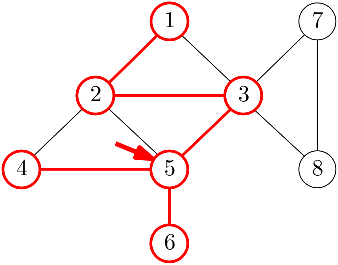

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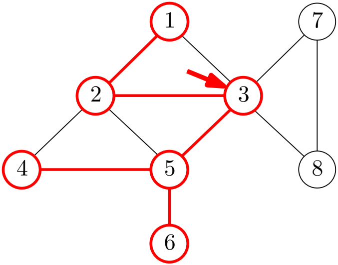

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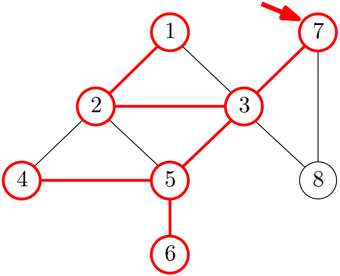

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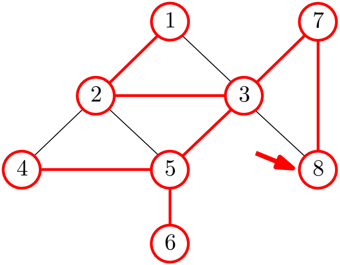


Implementing DFS using Recurrsion

DFS(s)

```
mark all vertices as "unvisited"
recursive-DFS(s)
```

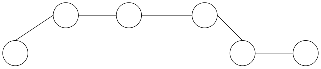
recursive-DFS(v)

```
mark v as "visited"
for all neighbors u of v do
    if u is unvisited then recursive-DFS(u)
```

Outline

Path Graph (or Linear Graph)

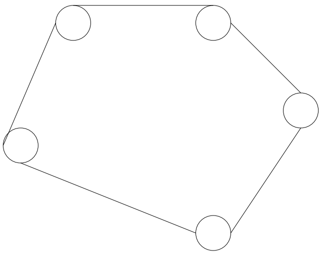
Def. An undirected graph $G = (V, E)$ is a path if the vertices can be listed in an order $\{v_1, v_2, ..., v_n\}$ such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, ..., n - 1$.



- Path graphs are connected graphs.

Cycle Graph (or Circular Graph)

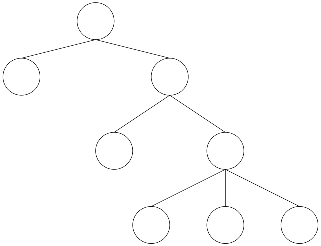
Def. An undirected graph $G = (V, E)$ is a cycle if its vertices can be listed in an order $v_1, v_2, ..., v_n$ such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, ..., n-1$, plus the edge $\{v_n, v_1\}$.



- The degree of all vertices is 2.

Tree

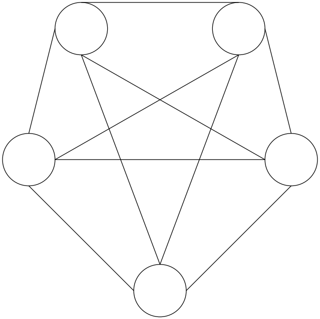
Def. An undirected graph $G = (V, E)$ is a tree if any two vertices are connected by exactly one path. Or the graph is a connected acyclic graph.



- Most important type of special graphs: most computational problems are easier to solve on trees or lines.

Complete Graph

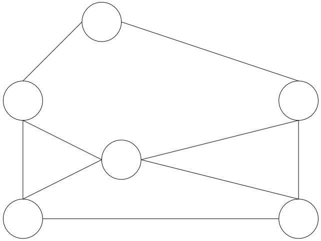
Def. An undirected graph $G = (V, E)$ is a complete graph if each pair of vertices is joined by an edge.



- A complete graph contains all possible edges.

Planar Graph

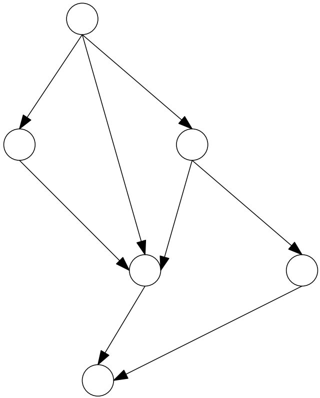
Def. An undirected graph $G = (V, E)$ is a planar graph if its vertices and edges can be drawn in a plane such that no two of the edges intersect.



- Most computational problems have good solutions in a planar graph.

Directed Acyclic Graph (DAG)

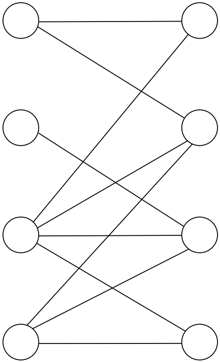
Def. A directed graph $G = (V, E)$ is a directed acyclic graph if it is a directed graph with no directed cycles



- DAG is equivalent to a partial ordering of nodes.

Bipartite Graph

Def. An undirected graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, either $u \in L, v \in R$ or $v \in L, u \in R$.



Outline

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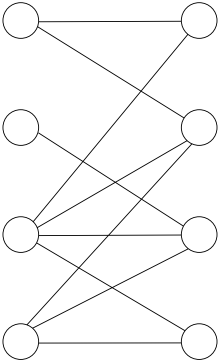
Testing Bipartiteness: Applications of BFS

Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, either $u \in L, v \in R$ or $v \in L, u \in R$.



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- Taking an arbitrary vertex $s \in V$

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- Report "not a bipartite graph" if contradiction was found

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- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
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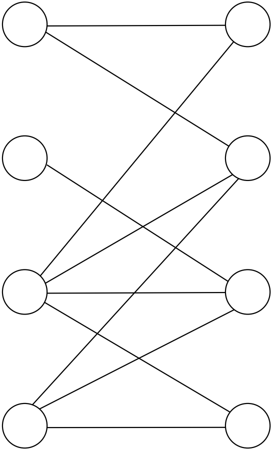
- $\cdots$
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness



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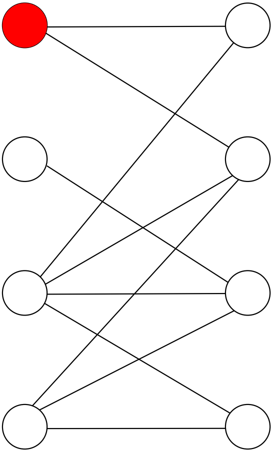

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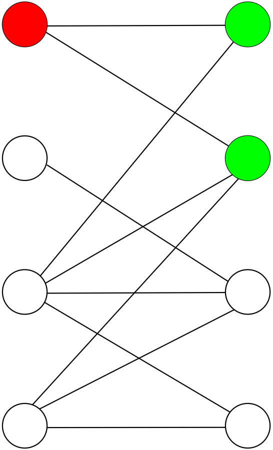

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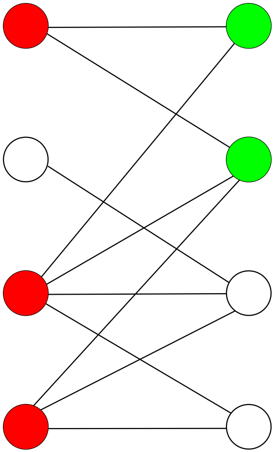

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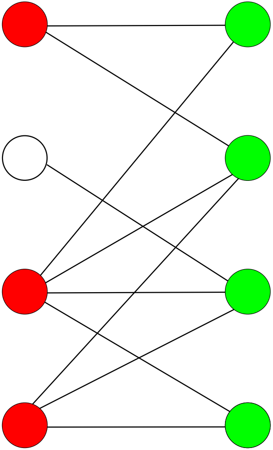

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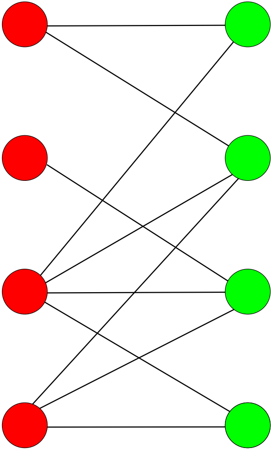

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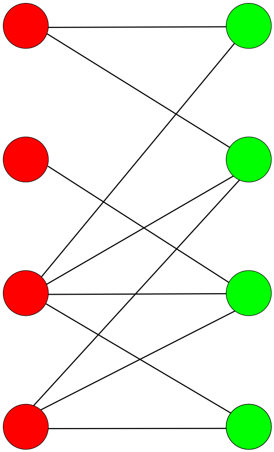

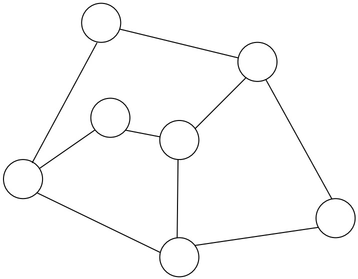

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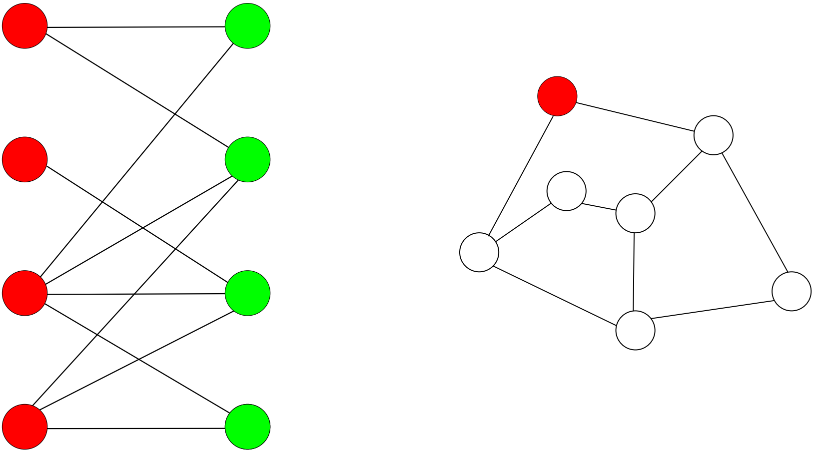

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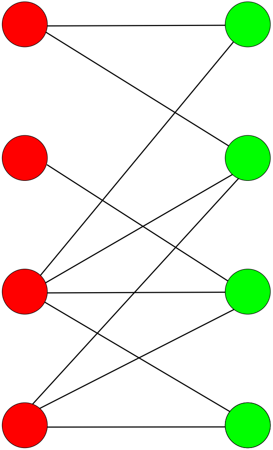

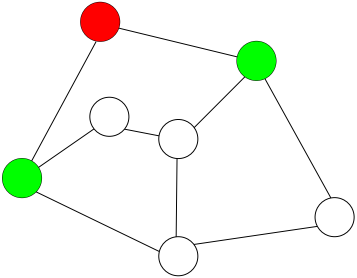

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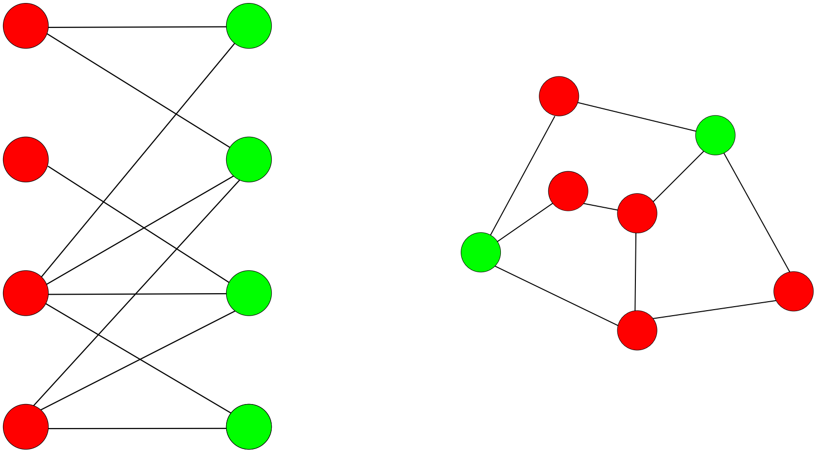

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