extract_min()

```
$ret \leftarrow A[1]$
$A[1] \leftarrow A[s]$
$p[A[1]] \leftarrow 1$
$s \leftarrow s - 1$
if $s \geq 1$ then
    heapify_down$(1)$
return $ret$
```

decrease_key(v, key_val

```
$key[v] \leftarrow key\_value$
heapify-up$(p[v])$
```

heapify-down(i)

```
while $2i \leq s$ do
    if $2i = s$ or $key[A[2i]] \leq key[A[2i+1]]$ then
        $j \leftarrow 2i$
    else
        $j \leftarrow 2i + 1$
    if $key[A[j]] < key[A[i]]$ then
        swap $A[i]$ and $A[j]$
        $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
        $i \leftarrow j$
    else break
```

- Running time of heapify_up and heapify_down: $O(\lg n)$

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data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

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Outline

1. Toy Example: Box Packing
2. Interval Scheduling
   - Interval Partitioning
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
6. Exercise Problems

Encoding Letters Using Bits

- 8 letters a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per letter

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \to 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?

Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.

Q: What is the issue with the following encoding scheme?

- a: 0 b: 1 c: 00

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A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

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A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

Solution

Use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of letters is a function $\gamma : S \to \{0, 1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

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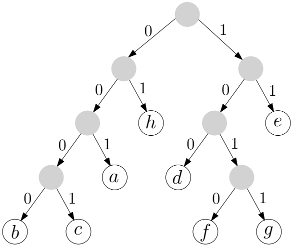
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



Prefix Codes Guarantee Unique Decoding

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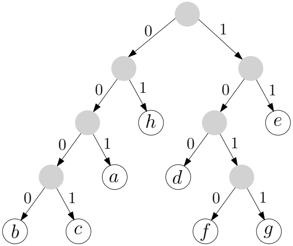

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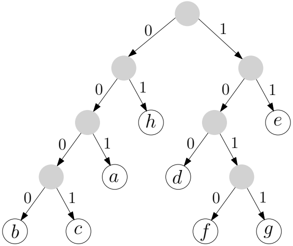

- 0001001100000001011110100001001

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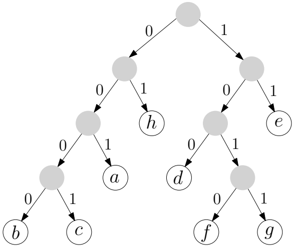

- 0001/0011000000010111101000001001
- c

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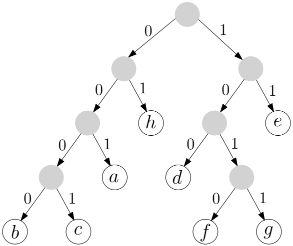

- 0001/001/100000001011110100001001
- ca

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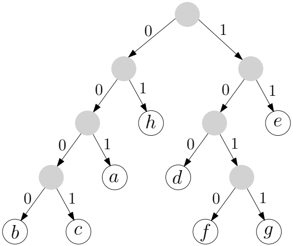

- 0001/001/100/0000010111110100001001
- cad

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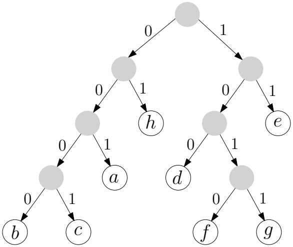

- 0001/001/100/0000/01011110100001001
- cadb

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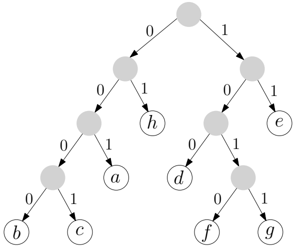

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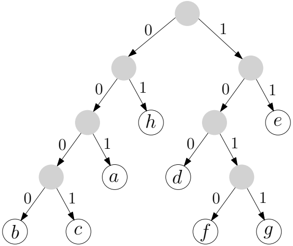

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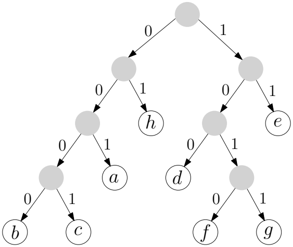

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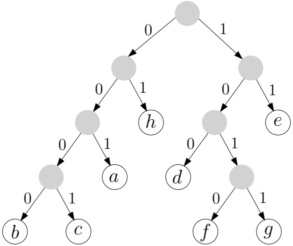

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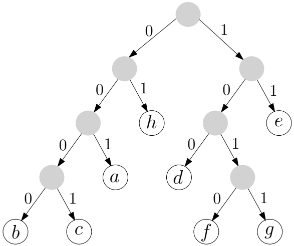

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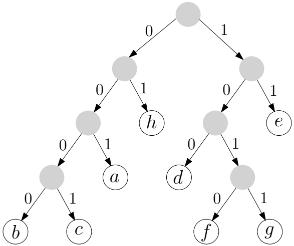

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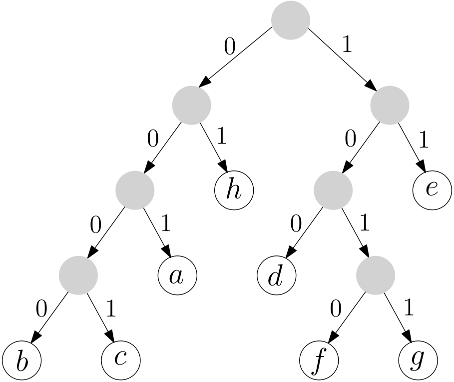
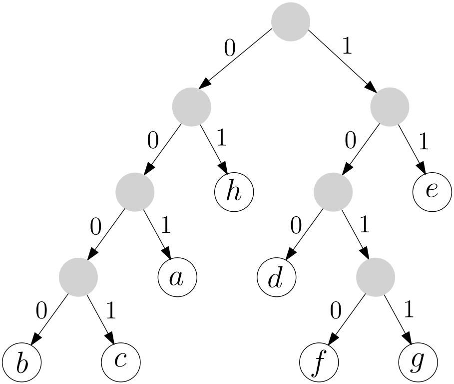
- cadbhhefca

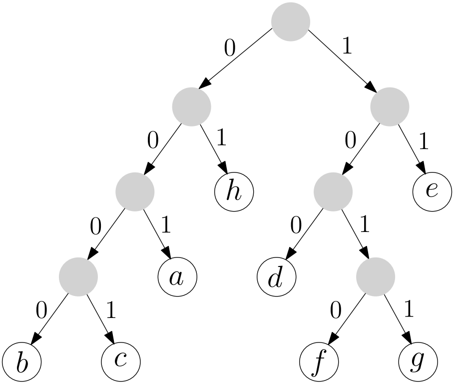


Properties of Encoding Tree



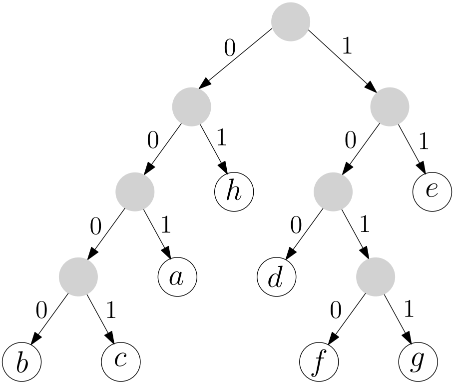
Properties of Encoding Tree

- Rooted binary tree



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1



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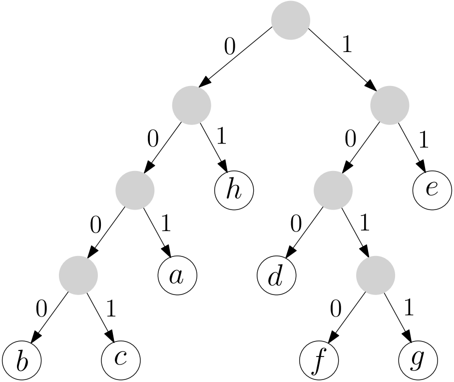

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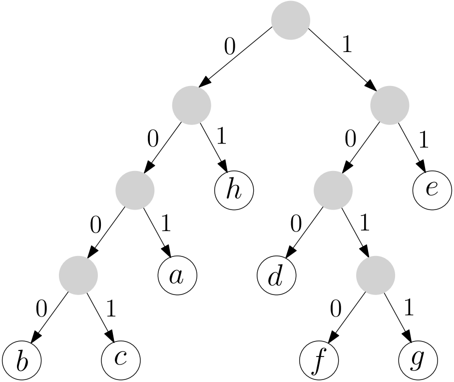

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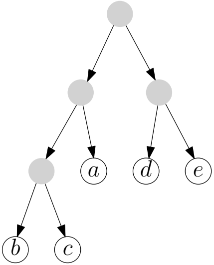
Best Prefix Codes

Input: frequencies of letters in a message

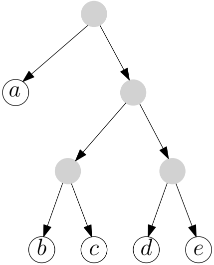
Output: prefix coding scheme with the shortest encoding for the message

example

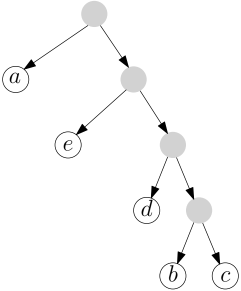
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



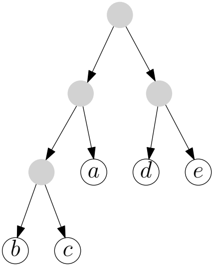
scheme 2



scheme 3

example

letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total = 89
scheme 2 length	1	3	3	3	3	total = 87
scheme 3 length	1	4	4	3	2	total = 84



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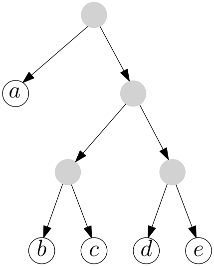
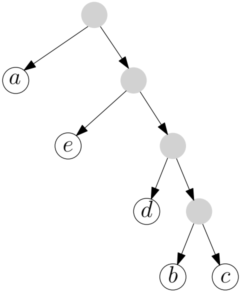


scheme 2



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- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.