Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

Q: What types of decisions should we make?

- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.
- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
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- There are two deepest leaves that are brothers
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Lemma

It is safe to make the two least frequent letters brothers.

best to put the two least frequent symbols here!
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers

**Lemma** It is safe to make the two least frequent letters brothers.
**Lemma** There is an optimum encoding tree, where the two least frequent letters are brothers.
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Q:  Is the residual problem another instance of the best prefix codes problem?
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**Q:** Is the residual problem another instance of the best prefix codes problem?

**A:** Yes, though it is not immediate to see why.
• $f_x$: the frequency of the letter $x$ in the support.
• $x_1$ and $x_2$: the two letters we decided to put together.
• $d_x$ the depth of letter $x$ in our output encoding tree.

\[
\begin{align*}
\sum_{x \in S} f_x d_x &= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
&= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
\end{align*}
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Def: $f_{x'} = f_{x_1} + f_{x_2}$
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\]
\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'} (d_{x'} + 1)
\]
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= \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x + f_{x'}
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\]

Def: $f_{x'} = f_{x_1} + f_{x_2}$
In order to minimize
\[ \sum_{x \in S} f_x d_x, \]
we need to minimize
\[ \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x, \]
subject to that \( d \) is the depth function for an encoding tree of
\( S \setminus \{x_1, x_2\} \).

This is exactly the best prefix codes problem, with letters
\( S \setminus \{x_1, x_2\} \cup \{x'\} \) and frequency vector \( f \)!
Example

A 27  B 15  C 11  D 9  E 8  F 5
Example
Example

A \hspace{2cm} B \hspace{2cm} C \hspace{2cm} D \hspace{2cm} E \hspace{2cm} F

27 \hspace{2cm} 15 \hspace{2cm} 11 \hspace{2cm} 20 \hspace{2cm} 8 \hspace{2cm} 13

D \quad E

9 \quad 5
Example
Example
Example
Example

Diagram of a tree with nodes labeled A, B, C, D, E, and F. Numbers represent values or indices at each node and branch.
Example

A: 00
B: 10
C: 010
D: 011
E: 110
F: 111
**Def.** The codes given by the greedy algorithm is called the **Huffman codes**.
Def. The codes given the greedy algorithm is called the Huffman codes.

Huffman($S, f$)

1: while $|S| > 1$ do
2: let $x_1, x_2$ be the two letters with the smallest $f$ values
3: introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
4: let $x_1$ and $x_2$ be the two children of $x'$
5: $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
6: return the tree constructed
Algorithm using Priority Queue

Huffman\((S, f)\)

1: \(Q \leftarrow \text{build-priority-queue}(S)\)
2: while \(Q.\text{size} > 1\) do
3: \(x_1 \leftarrow Q.\text{extract-min()}\)
4: \(x_2 \leftarrow Q.\text{extract-min()}\)
5: introduce a new letter \(x'\) and let \(f_{x'} = f_{x_1} + f_{x_2}\)
6: let \(x_1\) and \(x_2\) be the two children of \(x'\)
7: \(Q.\text{insert}(x', f_{x'})\)
8: return the tree constructed
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
   - Interval Partitioning
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
6. Exercise Problems
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a “reasonable” strategy
## Greedy Algorithm

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- Interval scheduling problem: schedule the job \( j^* \) with the earliest deadline
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- Offline Caching: evict the page that is used furthest in the future
- Huffman codes: make the two least frequent letters brothers
Summary for Greedy Algorithms

Analysis of Greedy Algorithm

- **Safety**: Prove that the reasonable strategy is “safe” (key)
- **Self-reduce**: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
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- **Self-reduce**: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem *(usually easy)*

**Def.** A strategy is “safe” if there is always an optimum solution that “agrees with” the decision made according to the strategy.
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
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Hu man codes: move the two least frequent letters to the deepest leaves.

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Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is “safe” (key)
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Interval scheduling problem: remove $j^\ast$ and the jobs it conflicts with
## Summary for Greedy Algorithms

### Analysis of Greedy Algorithm

- **Safety**: Prove that the reasonable strategy is “safe” (key)
- **Self-reduce**: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
Summary for Greedy Algorithms

Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is “safe” (key)
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one