- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

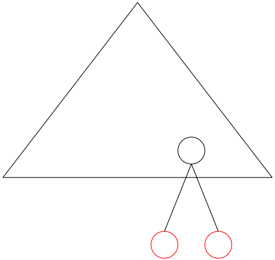
- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.

- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.

Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree



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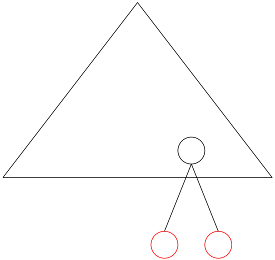
- Focus on the "structure" of the optimum encoding tree
- There are two deepest leaves that are brothers



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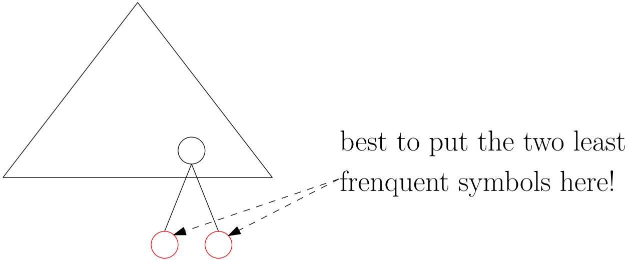

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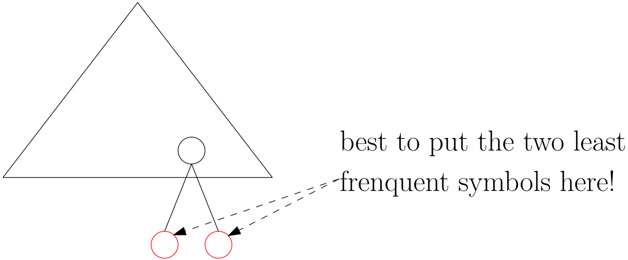

Lemma It is safe to make the two least frequent letters brothers.

Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

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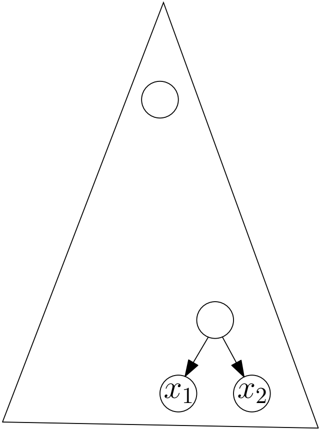
- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.

- f_x: the frequency of the letter x in the support.
- x_1 and x_2: the two letters we decided to put together.
- d_x the depth of letter x in our output encoding tree.



$$
\begin{aligned}
& \sum_{x \in S} f_x d_x \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
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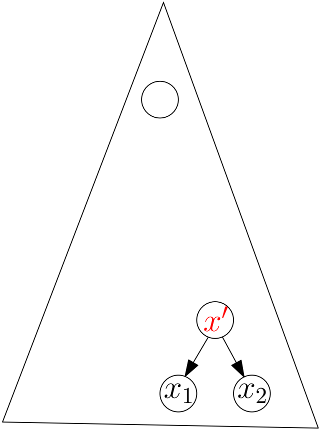

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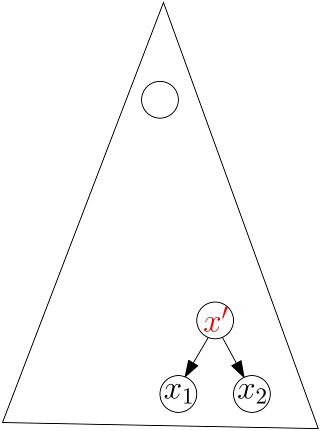

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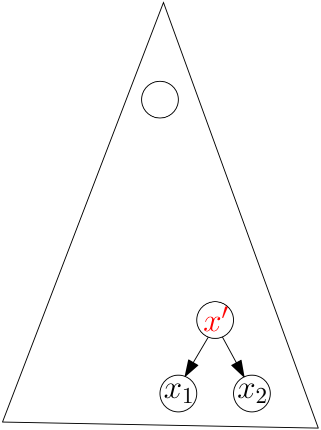

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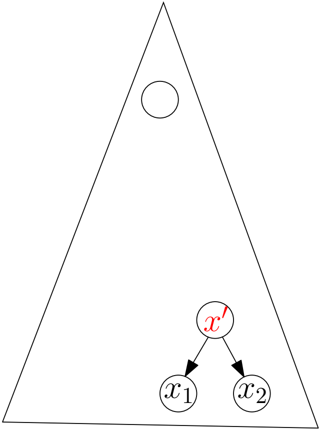

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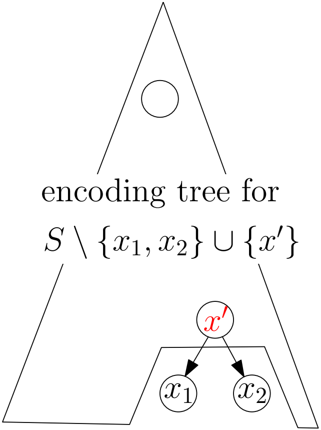

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Def: $f_{x'} = f_{x_1} + f_{x_2}$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Example

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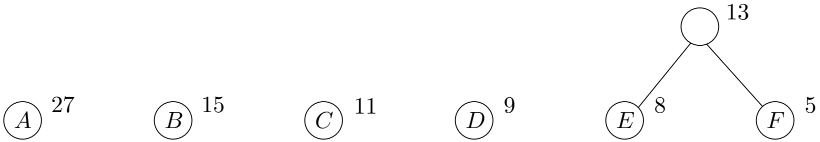

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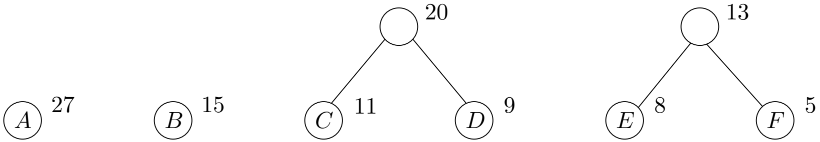

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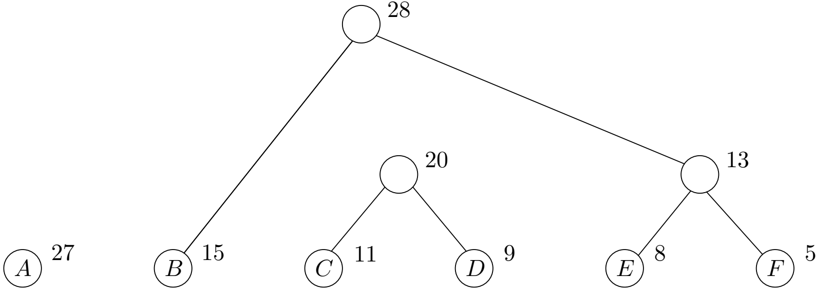

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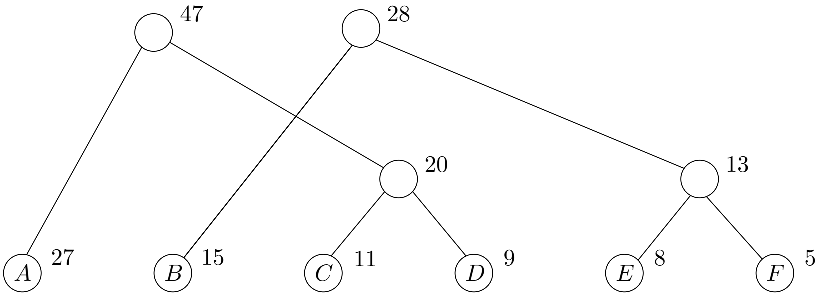

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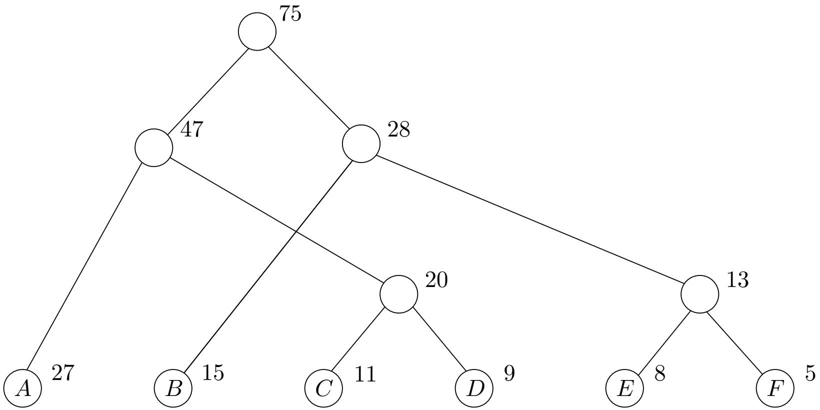

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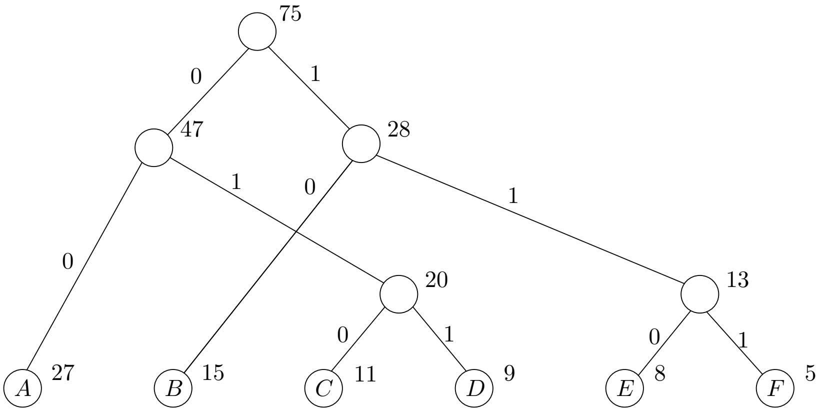

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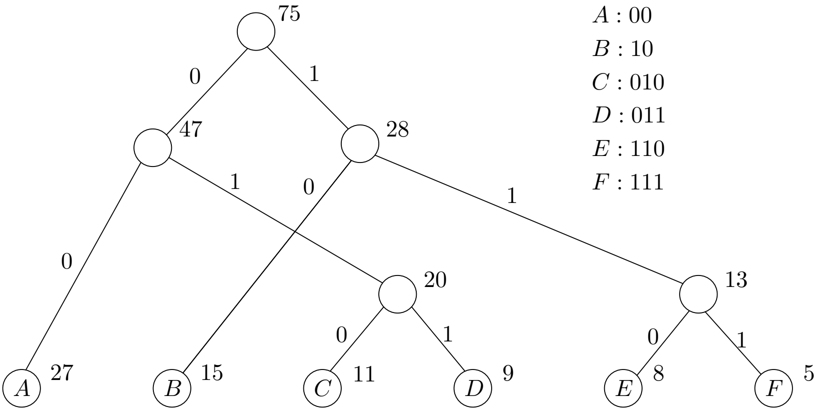

A : 00
B : 10
C : 010
D : 011
E : 110
F : 111

Def. The codes given the greedy algorithm is called the Huffman codes.

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Huffman(S, f)

```
while |S| > 1 do
    let x_1, x_2 be the two letters with the smallest f values
    introduce a new letter x' and let f_{x'} = f_{x_1} + f_{x_2}
    let x_1 and x_2 be the two children of x'
    S ← S \ {x_1, x_2} ∪ {x'}
return the tree constructed
```

Algorithm using Priority Queue

Huffman(S, f)

$Q \leftarrow$ build-priority-queue(S)
while $Q.\text{size} > 1$ **do**
    $x_1 \leftarrow Q.\text{extract-min}()$
    $x_2 \leftarrow Q.\text{extract-min}()$
    introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
    let x_1 and x_2 be the two children of x'
    $Q.\text{insert}(x', f_{x'})$
return the tree constructed

Outline

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Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

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- Offline caching: trivial
- Huffman codes: merge two letters into one