Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

```
A ← {1, 2, ⋯ , n}, S ← ∅
while A ≠ ∅ do
    j ← arg min_{j'∈A} f_{j'}
    S ← S ∪ {j}; A ← {j' ∈ A : s_{j'} ≥ f_j}
return S
```

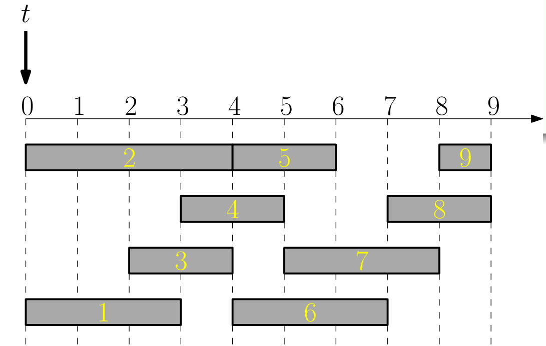
Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

sort jobs according to f values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of f_j **do**
    if $s_j \geq t$ **then**
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return S



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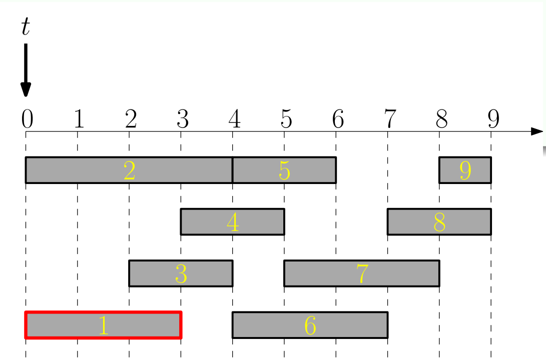

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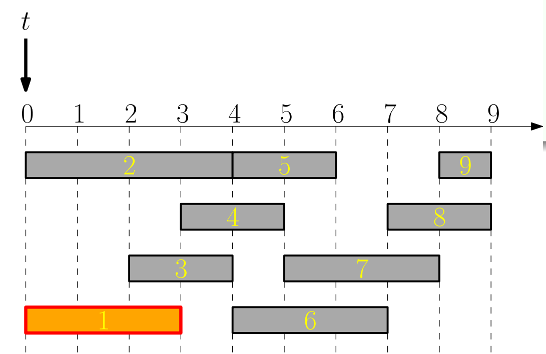

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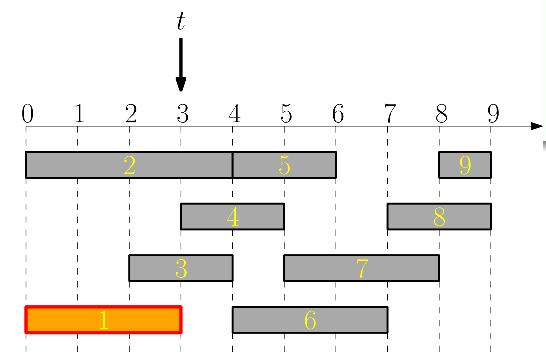

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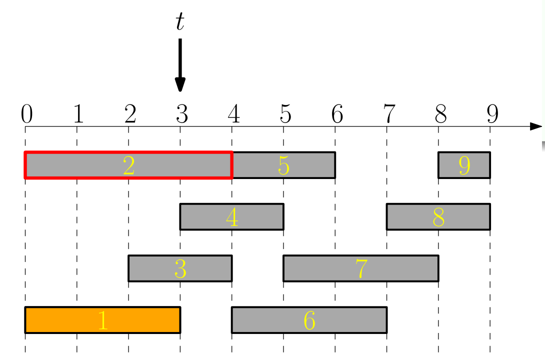

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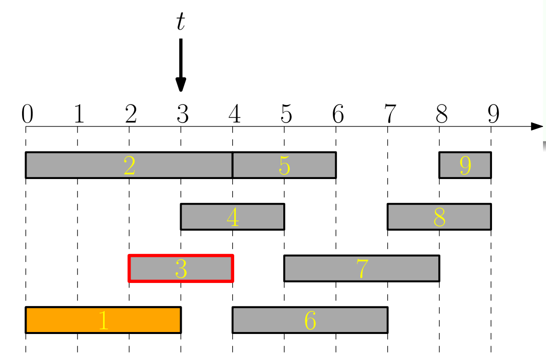

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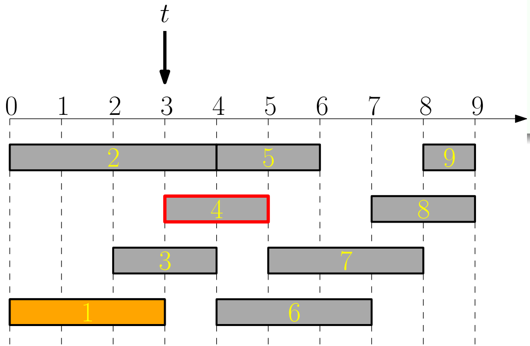

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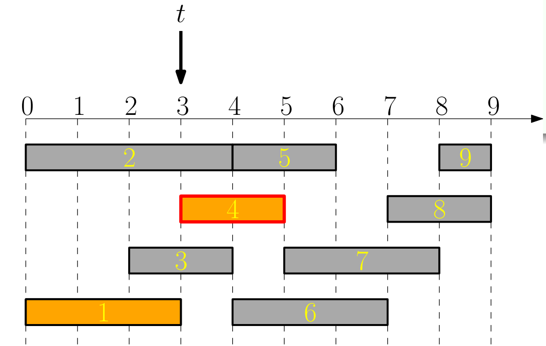

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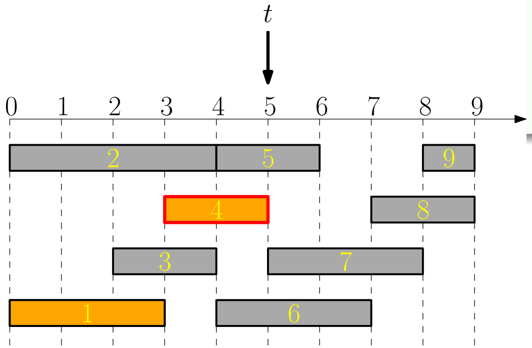

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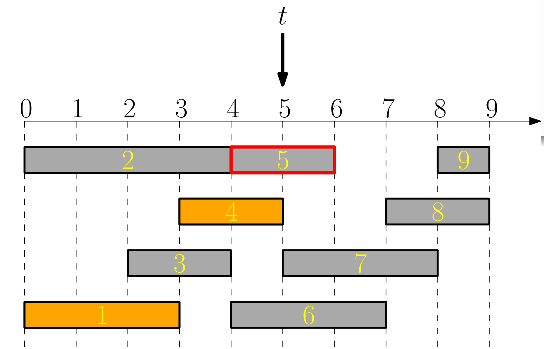

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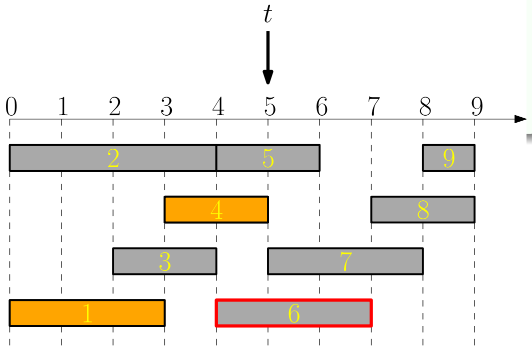

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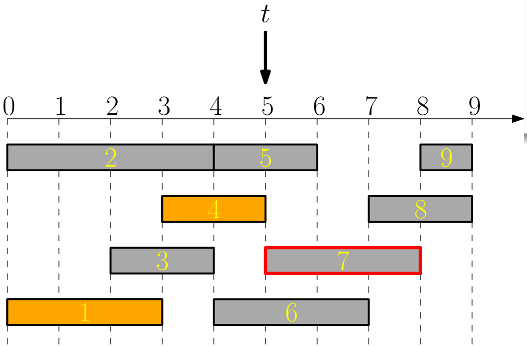

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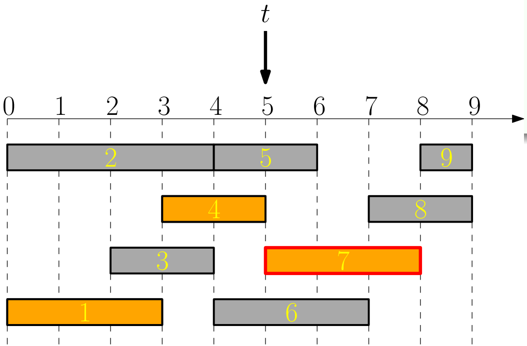

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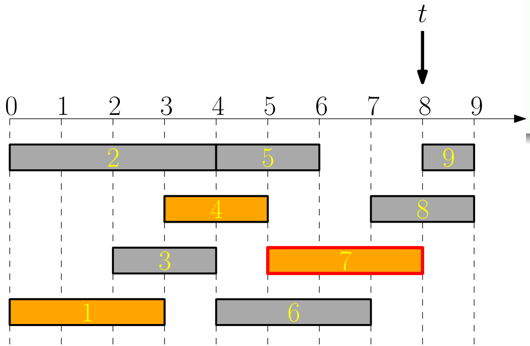

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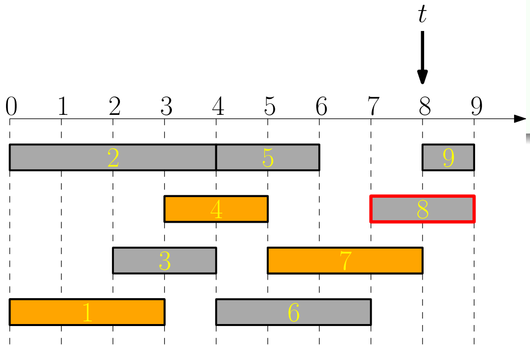

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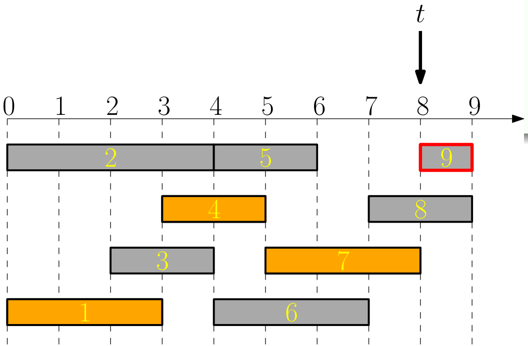

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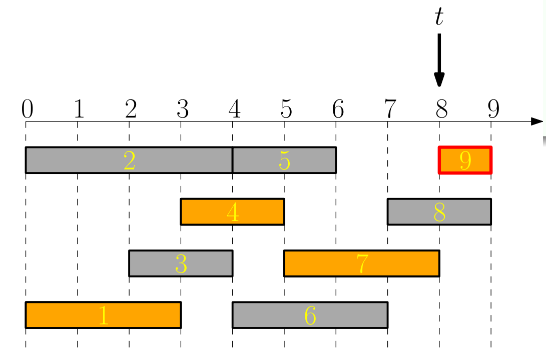

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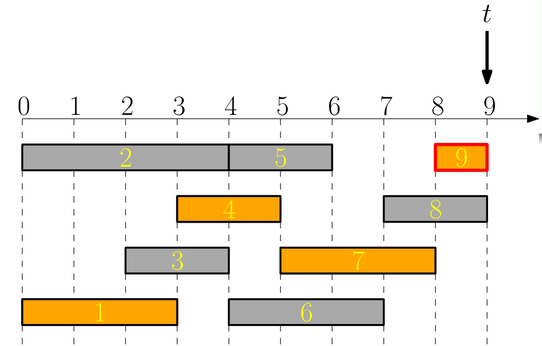


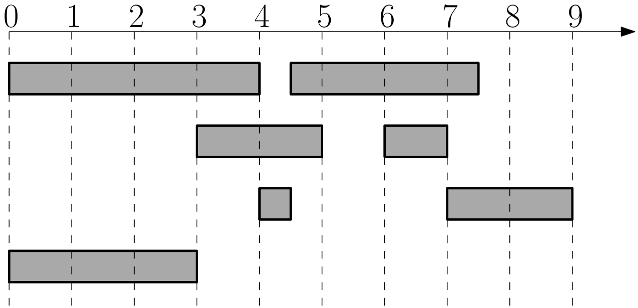
Outline

Interval Partitioning

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A minimum number of machines to schedule all jobs so that all jobs on a single machine are compatible.



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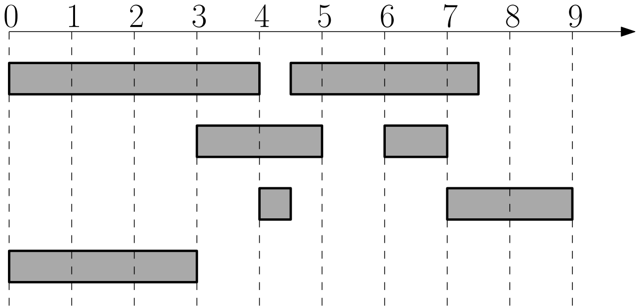

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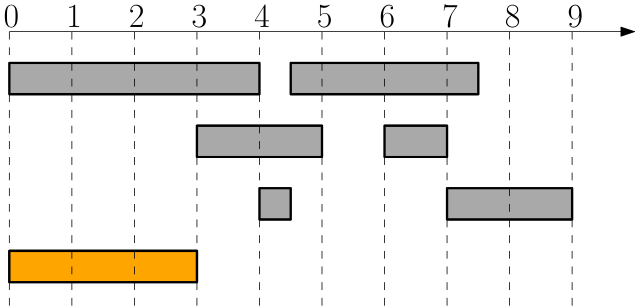

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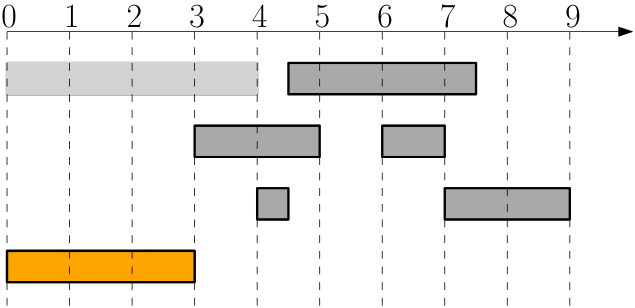

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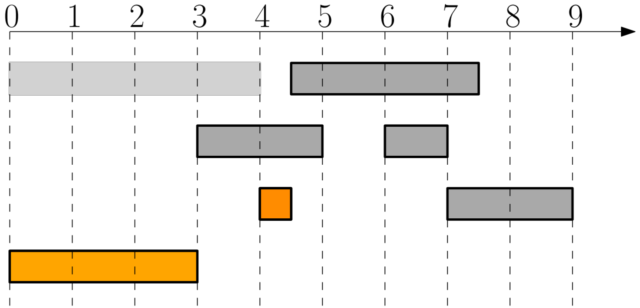

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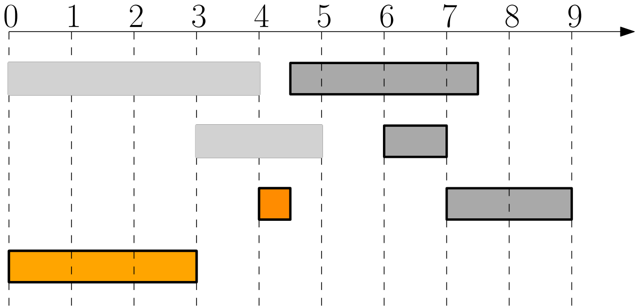

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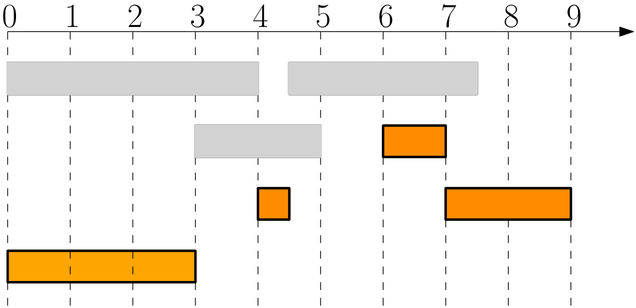

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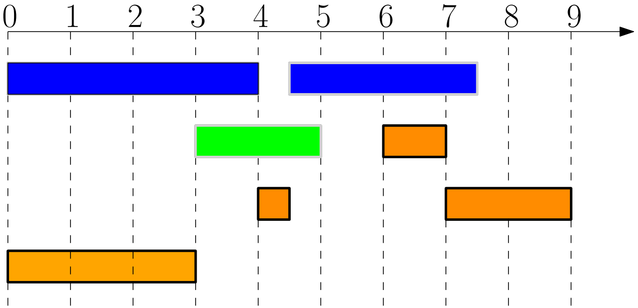

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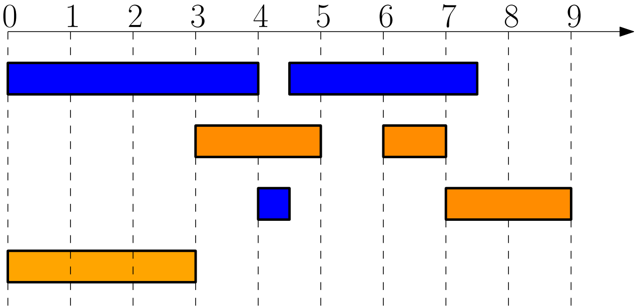

Greedy Algorithm for Interval Partitioning

Lemma It is safe to schedule the job j with the earliest starting time to a feasible machine: There exists an optimum solution where job j with the earliest starting time is scheduled first on a machine that is compatible with all jobs in that machine if applicable; otherwise, it can be scheduled by opening a new machine.

Proof.

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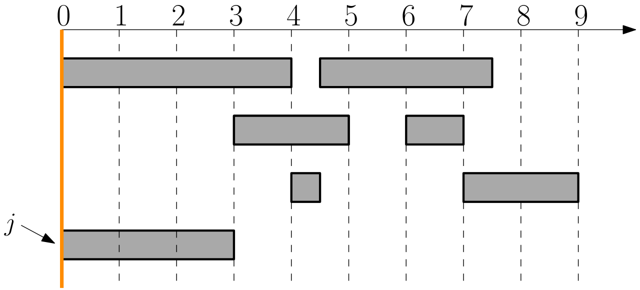
- Take an arbitrary optimum solution S
- If it schedules j to the chosen feasible machine i, done
- Otherwise, replace all the jobs scheduled to the machine i in S with j and its subsequent jobs to obtain another optimum schedule S'. □

Greedy Algorithm for Interval Partitioning

- What is the remaining task after we decided to schedule j?
- Is it another instance of interval partitioning problem?



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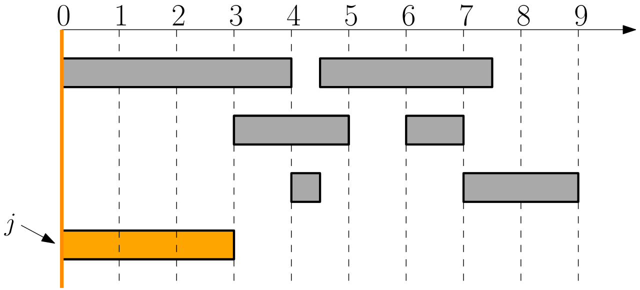

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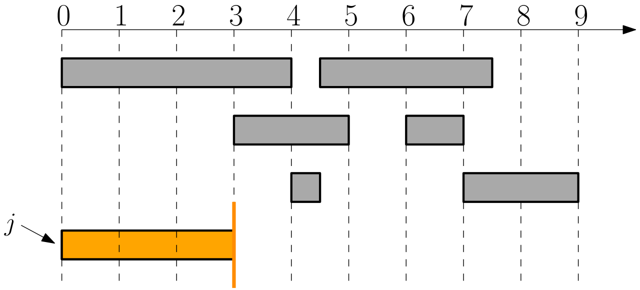


Greedy Algorithm for Interval Partitioning

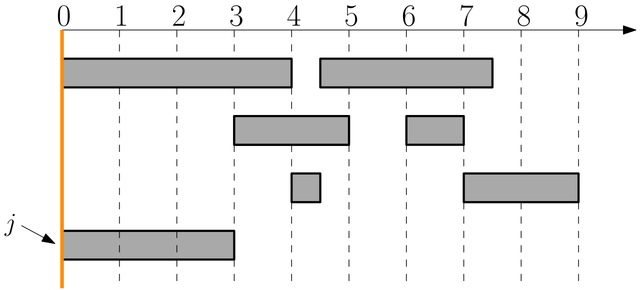
Partition(s, f, n)

```
A ← {1, 2, ⋯, n}, S ← {1}, t_1 = 0
while A ≠ ∅ do
    j ← argmin_{j'∈A} s_{j'}, S_j ← {i'}_{i'∈S, t_{i'} ≤ s_j}
    If S_j ≠ ∅, then schedule j to a machine i ∈ S_j and t_i = f_j
    Otherwise, schedule j to machine |S| + 1, S ← S ∪ {|S| + 1} and t_{|S|} = f_j
return S
```

In math notation:

- $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \{1\}$, $t_1 = 0$
- **while** $A \neq \emptyset$ **do**
  - $j \leftarrow \arg\min_{j' \in A} s_{j'}$, $S_j \leftarrow \{i'\}_{i' \in S, t_{i'} \leq s_j}$
  - If $S_j \neq \emptyset$, then schedule j to a machine $i \in S_j$ and $t_i = f_j$
  - Otherwise, schedule j to machine $|S| + 1$, $S \leftarrow S \cup \{|S| + 1\}$ and $t_{|S|} = f_j$
- **return** S

Greedy Algorithm for Interval Partitioning



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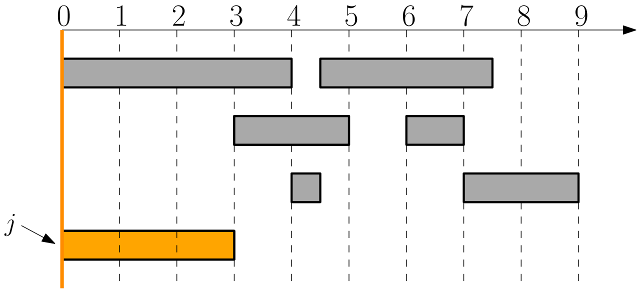

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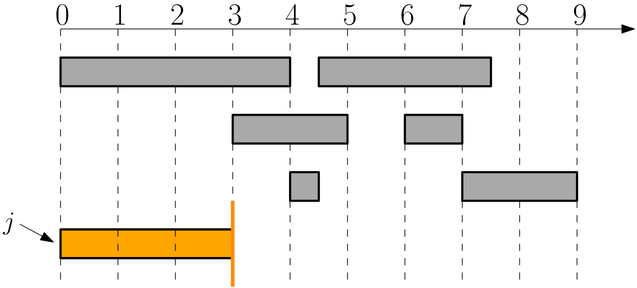

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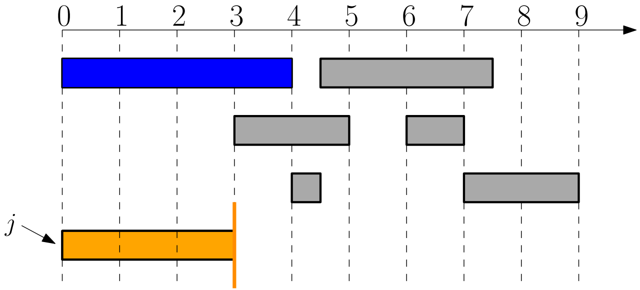

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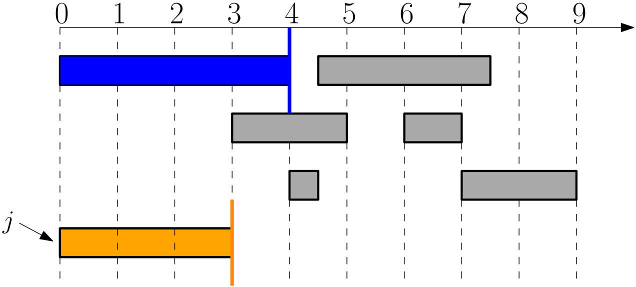

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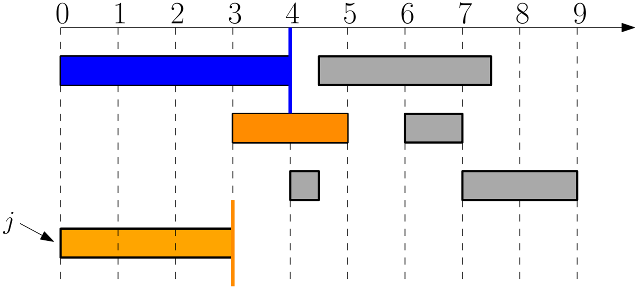

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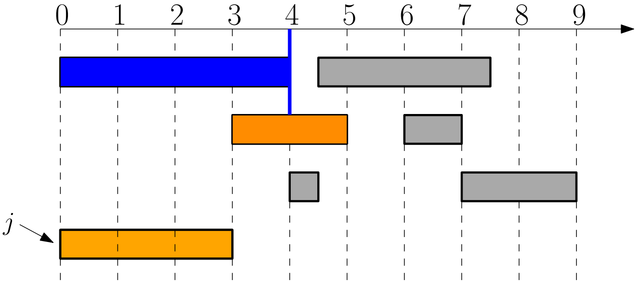

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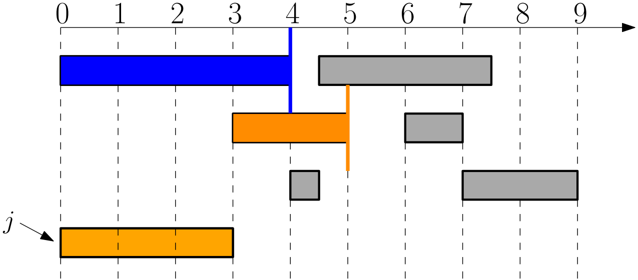

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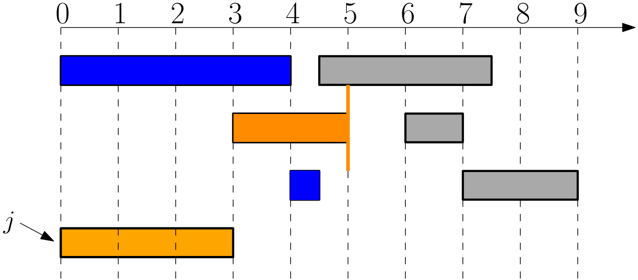

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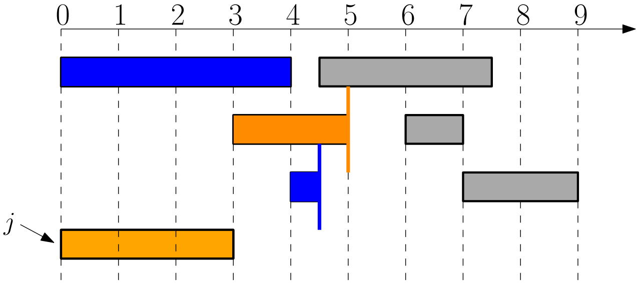

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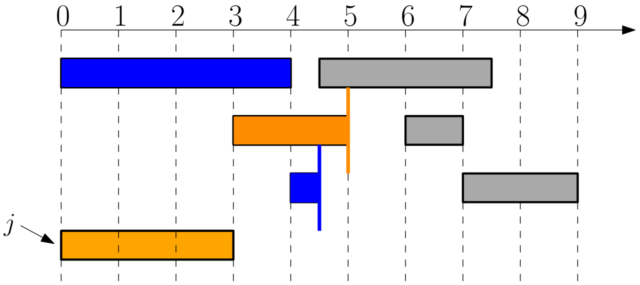

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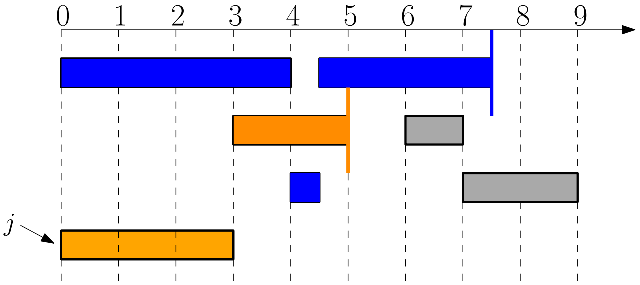

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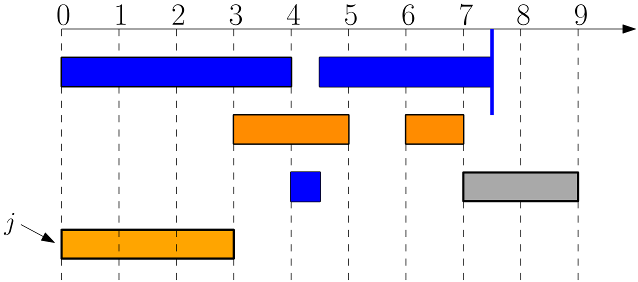

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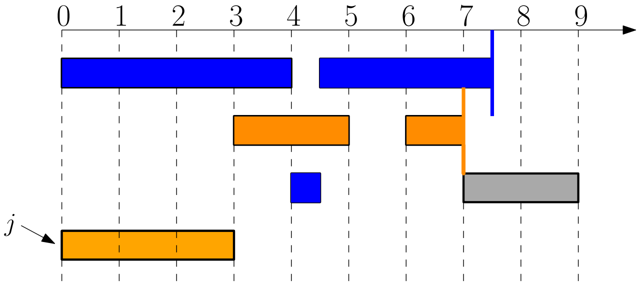

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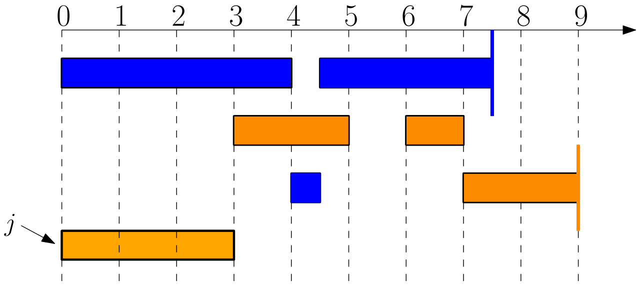

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Obs. Greedy algorithm never schedules two incompatible jobs in the same machine.

Why "Greedy algorithm" is optimal?

Theorem Greedy algorithm is optimal.

Proof.

- Let d be the number of machines that greedy algorithm used.

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- Observation: all these $d-1$ jobs starts earlier than s_j because we schedule the jobs in order of starting time. Thus, we have d jobs overlapping at time $s_j+\epsilon$. The jobs **depth** $\geq d$.

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- By the Observation in the previous slide, an optimal solution $\geq d$. Thus the greedy algorithm is optimal.

□

Greedy Algorithm for Interval Partitioning

Partition(s, f, n)

$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \{1\}$, $t_1 = 0$
while $A \neq \emptyset$ **do**
    $j \leftarrow \arg\min_{j' \in A} s_{j'}$, $S_j \leftarrow \{i'\}_{i' \in S, t_{i'} \leq s_j}$
    If $S_j \neq \emptyset$, then schedule j to a machine $i \in S_j$ and $t_i = f_j$
    Otherwise, schedule j to machine $|S| + 1$, $S \leftarrow S \cup \{|S| + 1\}$ and $t_{|S|} = f_j$
return S

Running time of algorithm?

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6: **return** S

Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time with Priority Queue.

Outline

Offline Caching

- Cache that can store k pages
- Sequence of page requests

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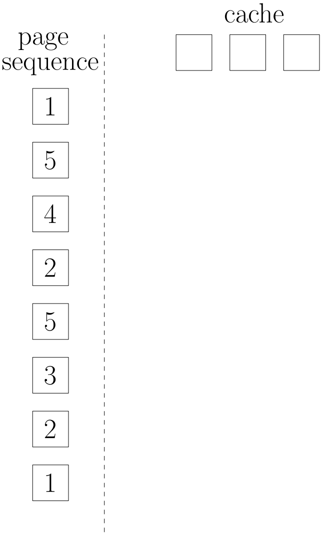

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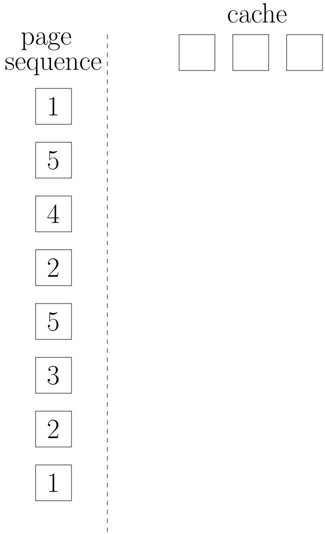

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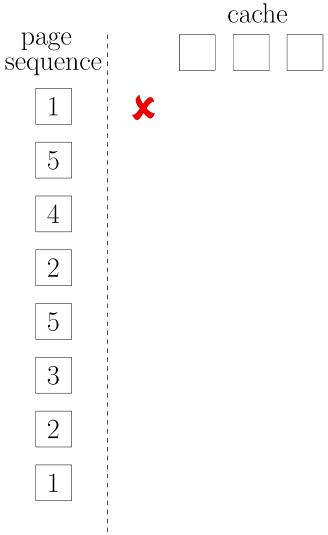

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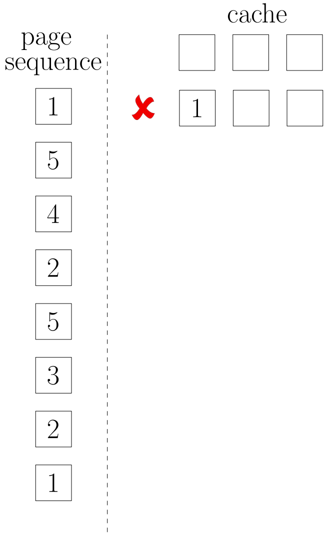

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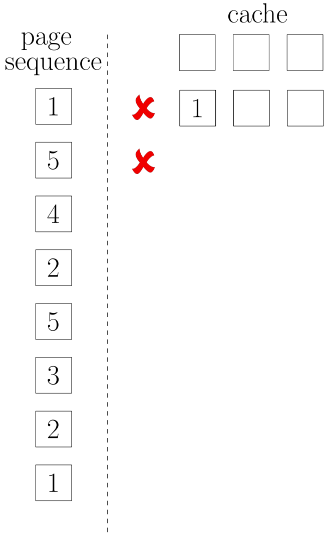

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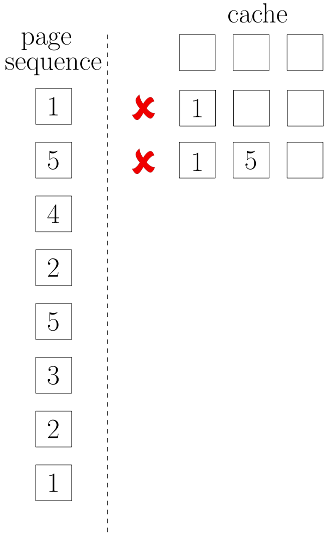

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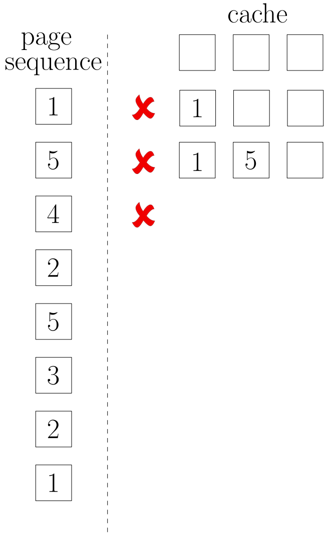

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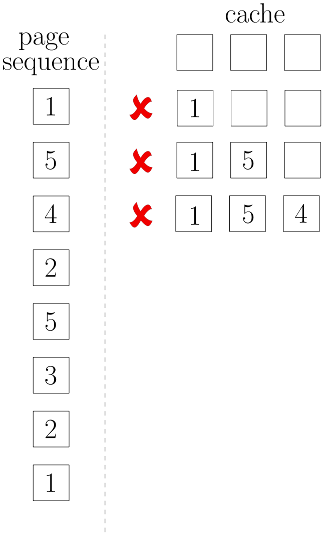

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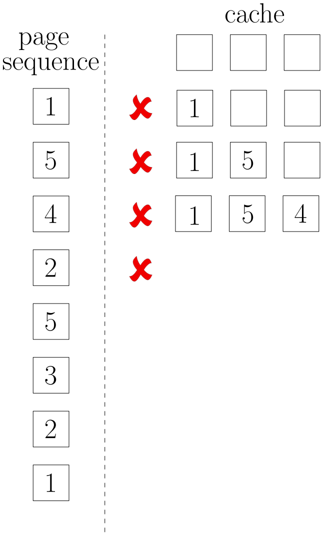

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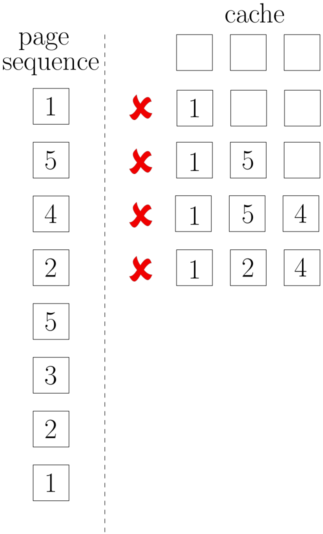

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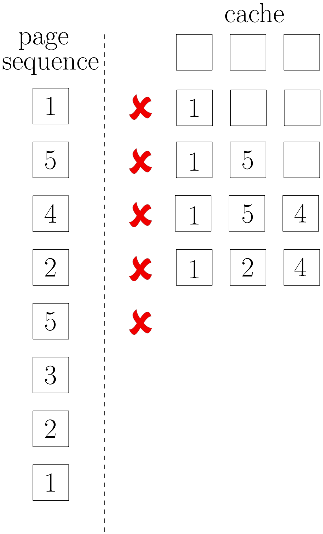

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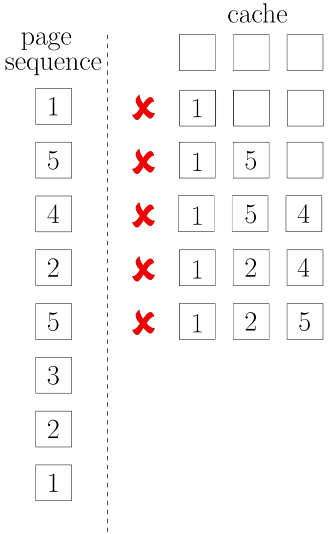

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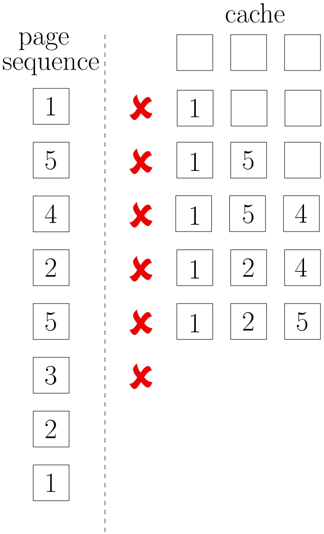

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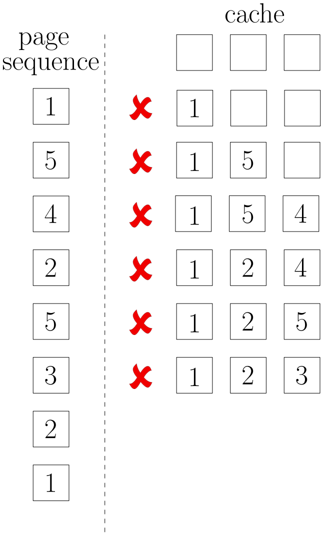

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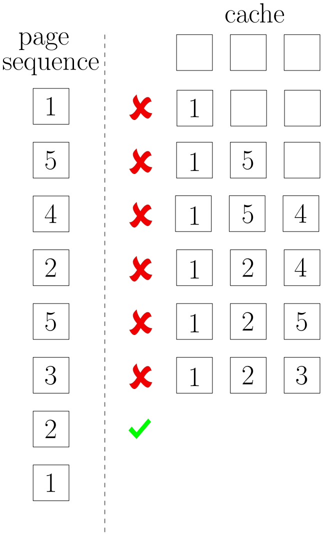

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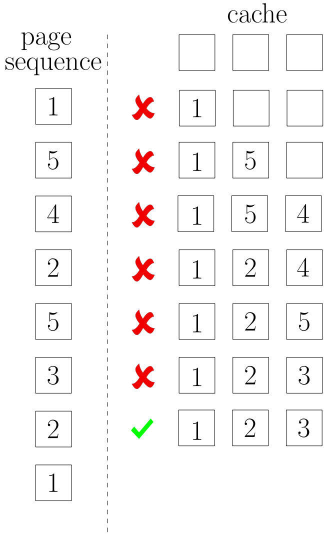

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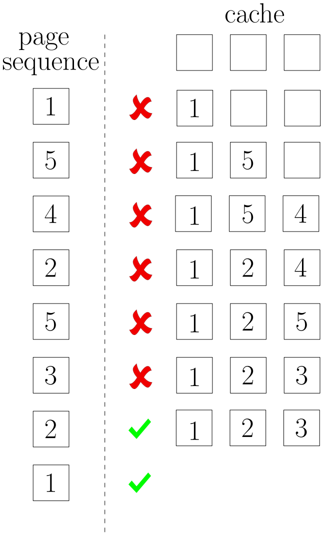

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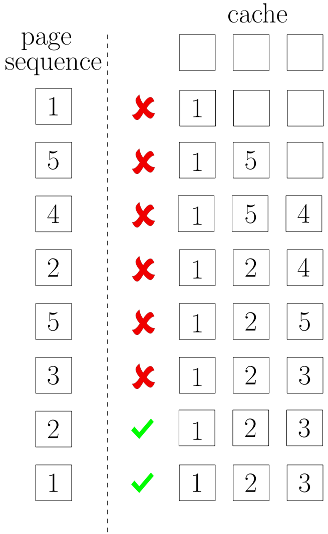

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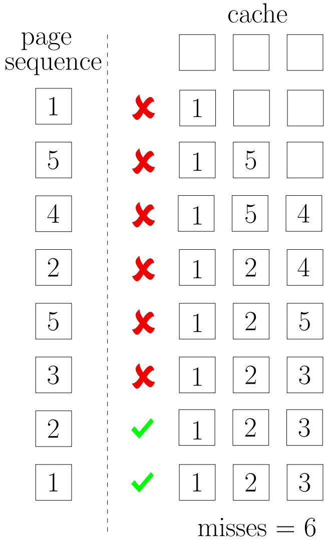

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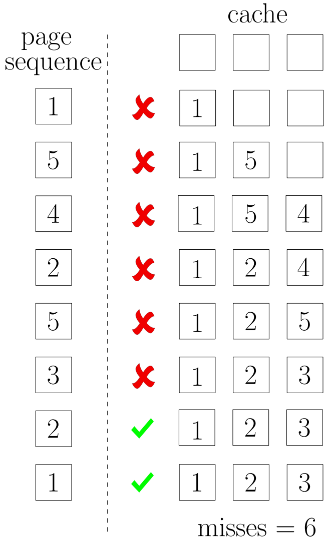
- Goal: minimize the number of cache misses.



A Better Solution for Example

page sequence		cache				cache		
1	✗	1			✗	1		
5	✗	1	5		✗	1	5	
4	✗	1	5	4	✗	1	5	4
2	✗	1	2	4	✗	1	5	2
5	✗	1	2	5	✓	1	5	2
3	✗	1	2	3	✗	1	3	2
2	✓	1	2	3	✓	1	3	2
1	✓	1	2	3	✓	1	3	2
		misses = 6				misses = 5		

Offline Caching Problem

Input: k : the size of cache

n : number of pages

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_T \in \{\mathrm{hit}, \mathrm{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

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A: Online caching

Q: Why do we study the offline caching problem?

A: Use the offline solution as a benchmark to measure the "competitive ratio" of online algorithms

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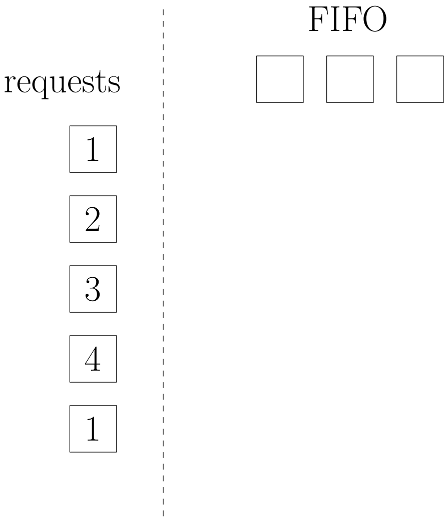
- All the above algorithms are not optimum!
- Indeed all the algorithms are "online", i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.

FIFO is not optimum



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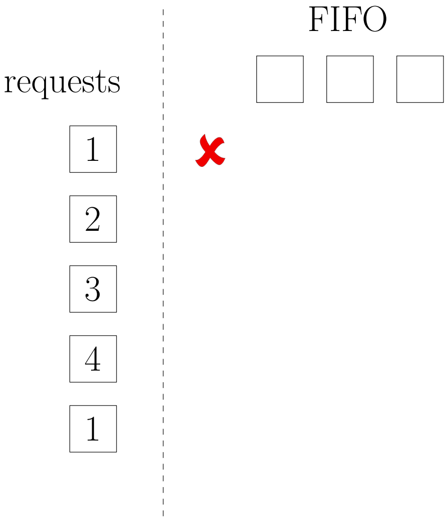

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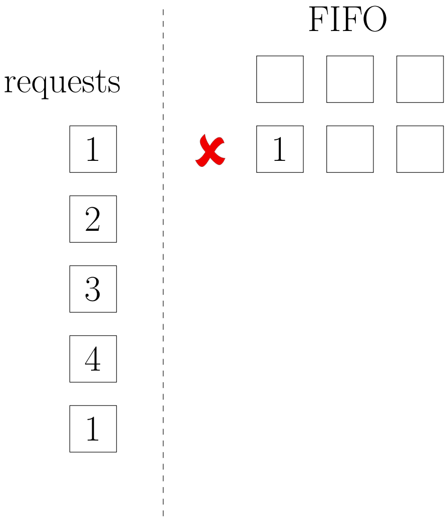

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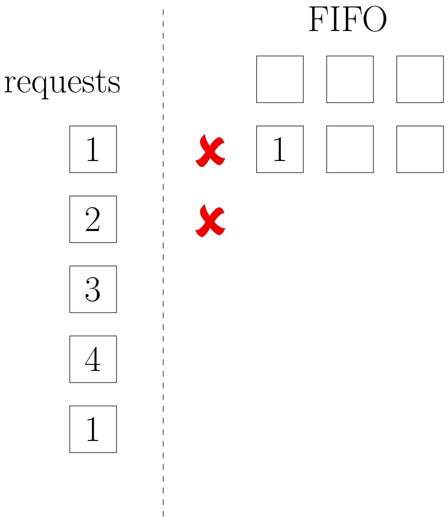

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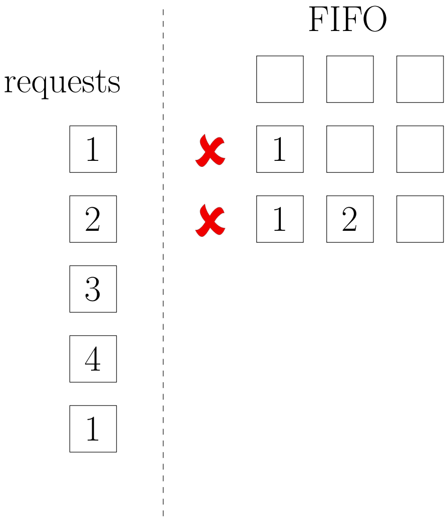

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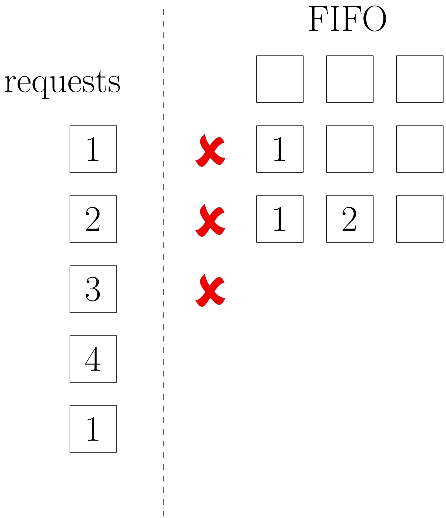

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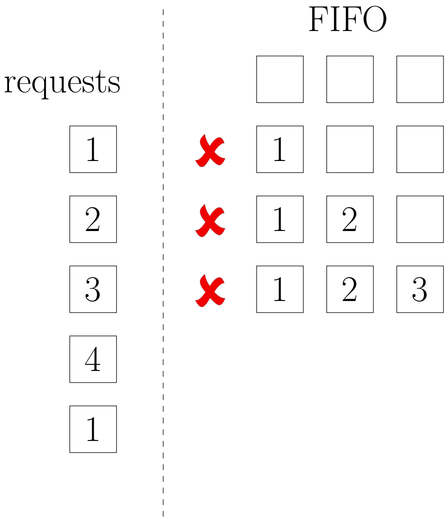

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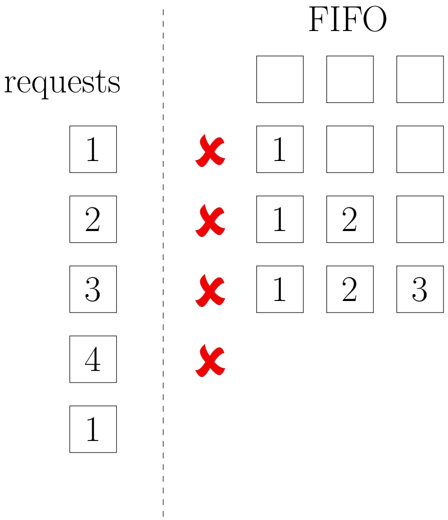

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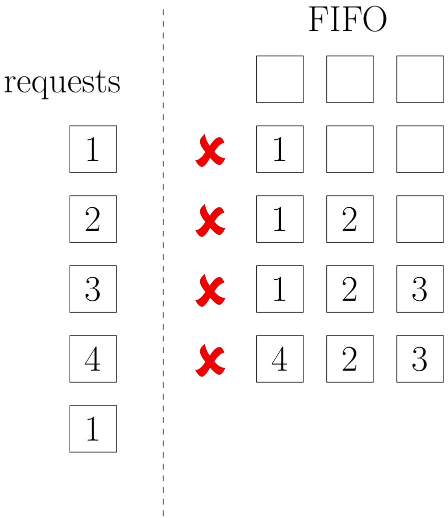

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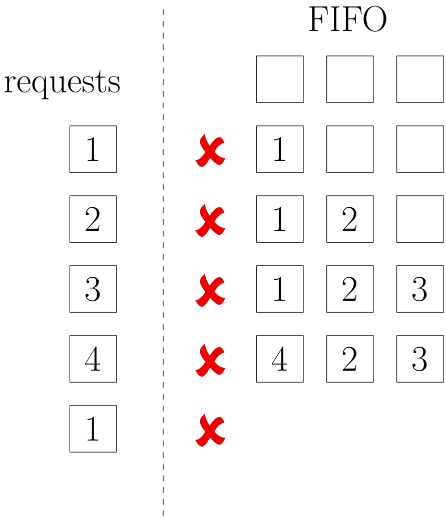

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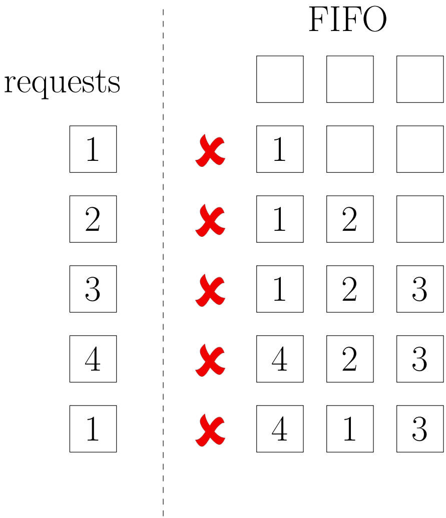


FIFO is not optimum

requests		FIFO		
1	✘	1		
2	✘	1	2	
3	✘	1	2	3
4	✘	4	2	3
1	✘	4	1	3

$\text{misses} = 5$

FIFO is not optimum

requests		FIFO				Furthest-in-Future		
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3

FIFO: misses $= 5$

Furthest-in-Future: misses $= 4$

Optimum Offline Caching

Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.

Furthest-in-Future (FF)

requests		FIFO				Furthest-in-Future		
1	✗	1			✗	1		
2	✗	1	2		✗	1	2	
3	✗	1	2	3	✗	1	2	3
4	✗	4	2	3	✗	1	4	3
1	✗	4	1	3	✓	1	4	3

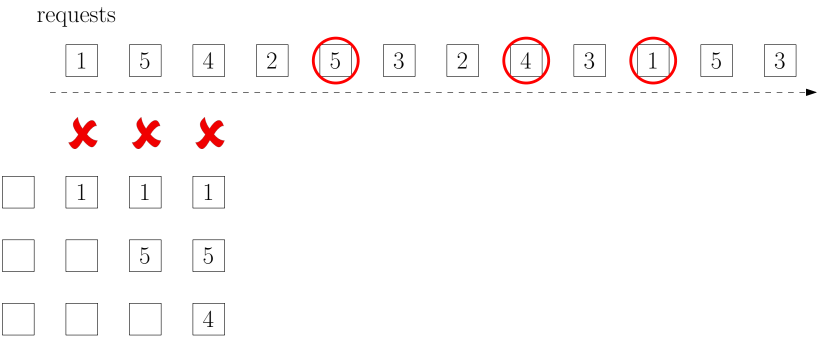
FIFO: misses = 5

Furthest-in-Future: misses = 4

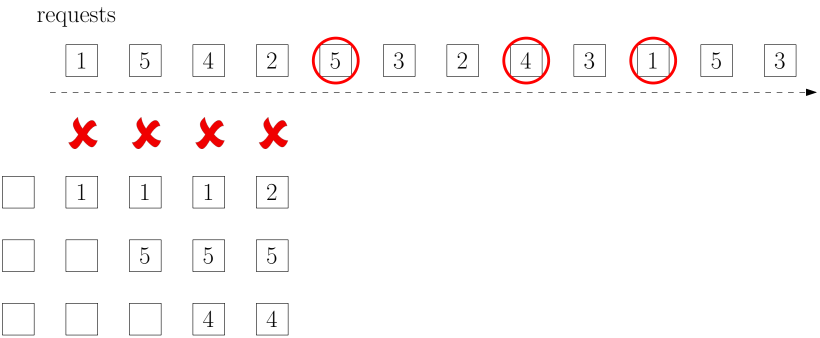
Example

Example

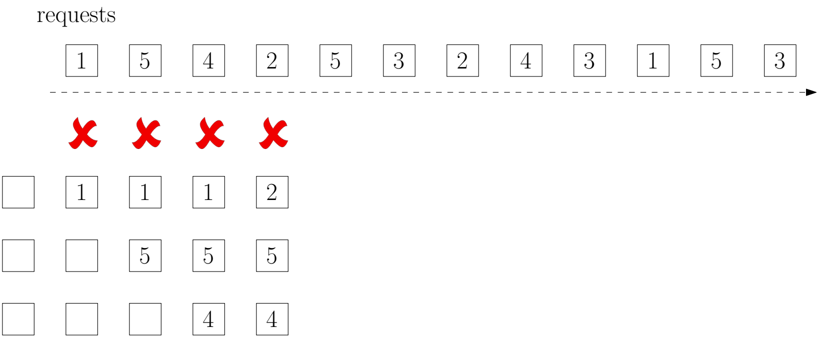
Example



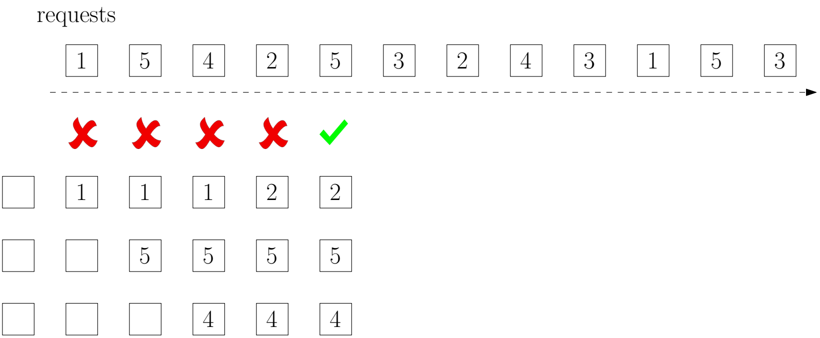
Example



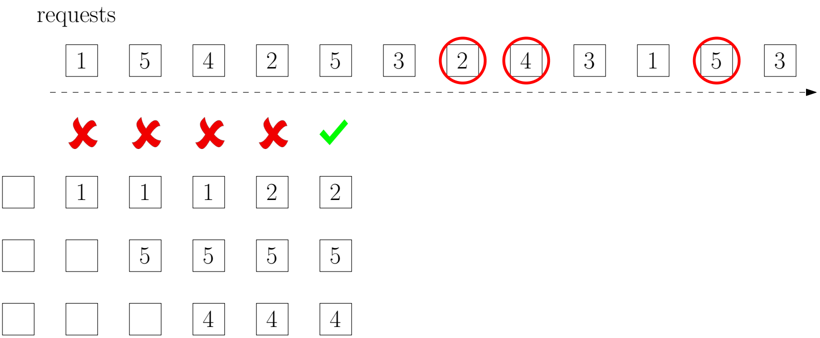
Example



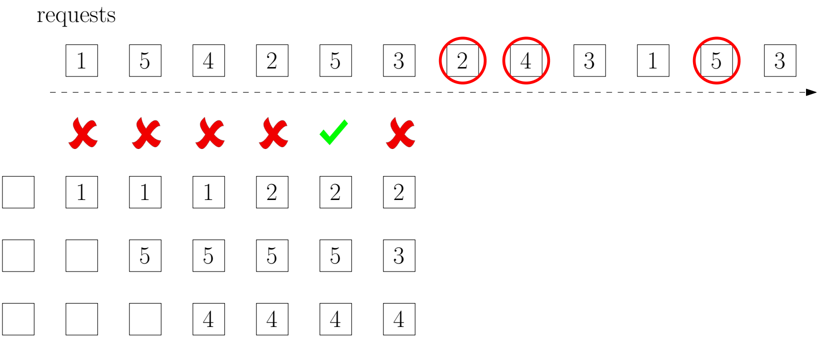
Example



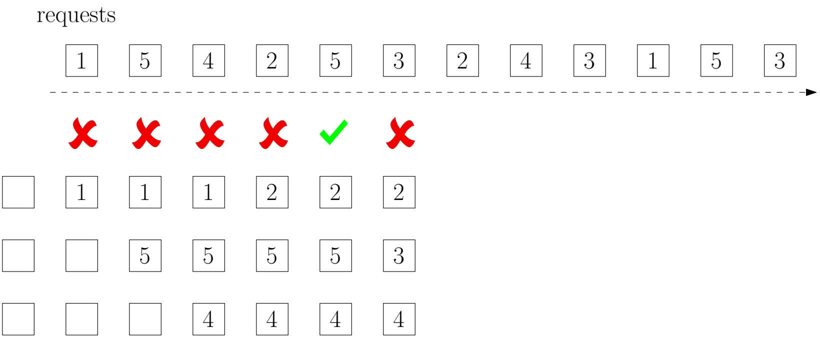
Example

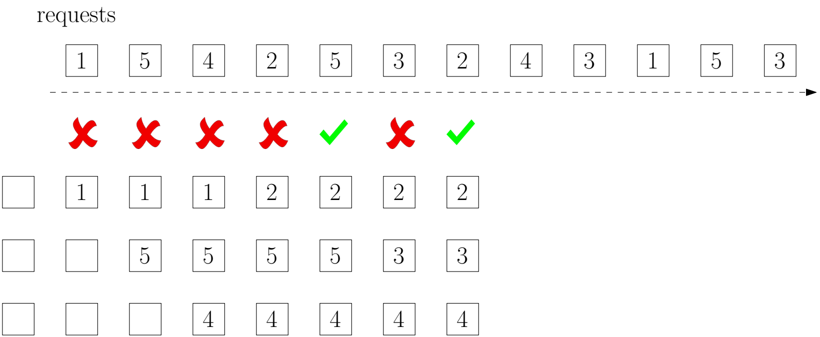


Example



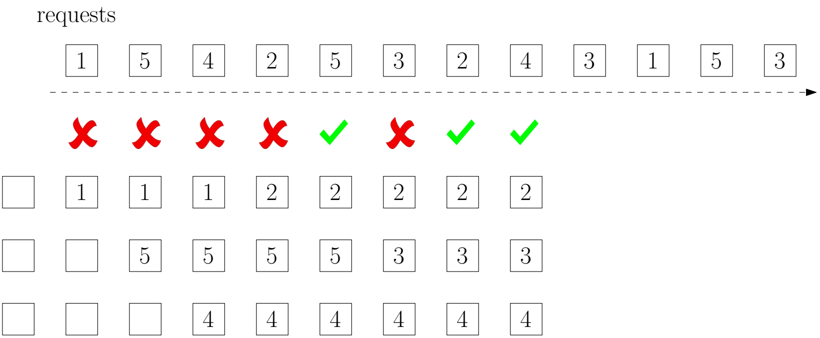
Example



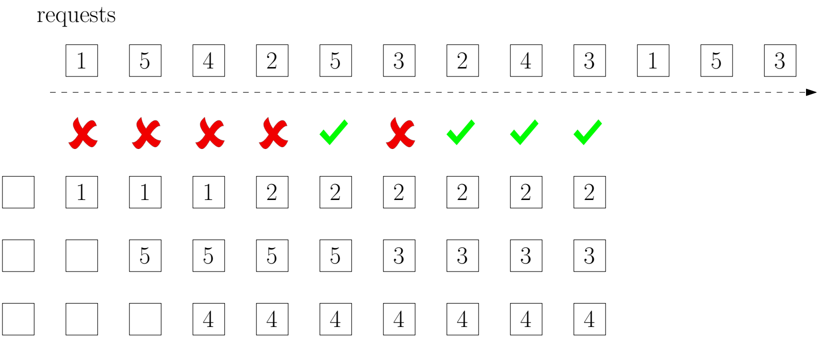


requests
1 5 4 2 5 3 2 4 3 1 5 3
1 1 1 2 2 2 2
5 5 5 5 3 3
4 4 4 4 4

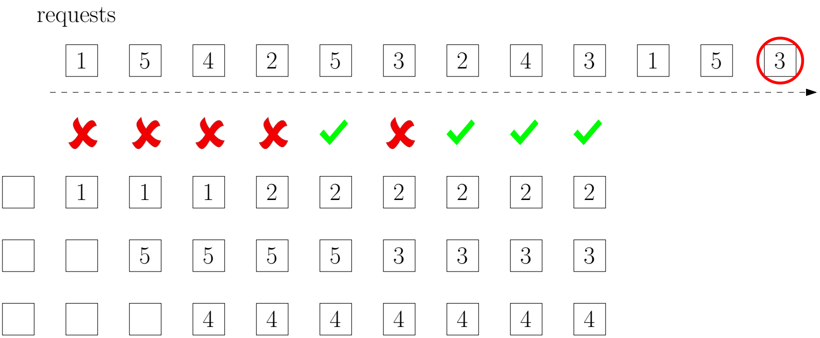
Example



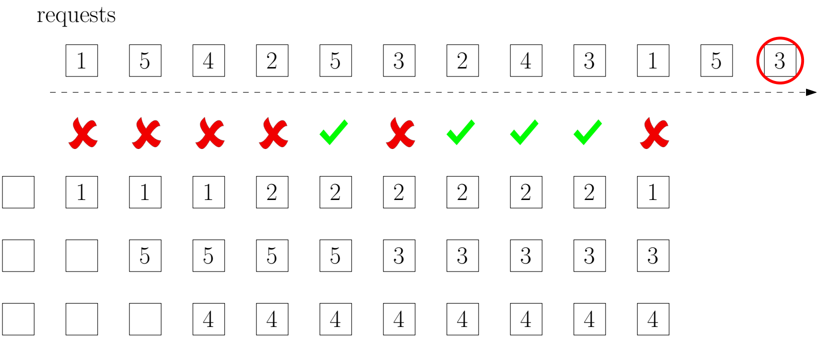
Example



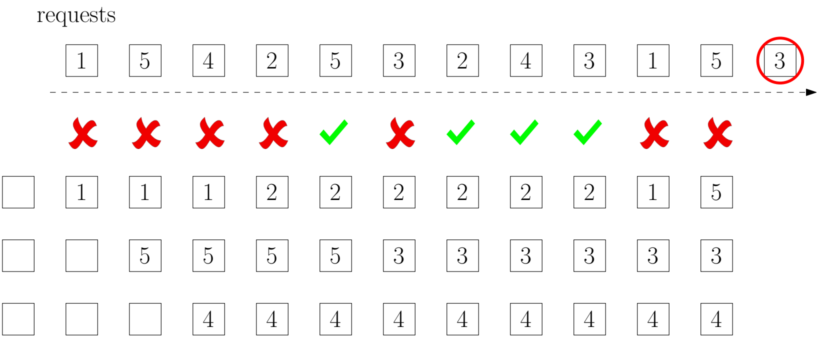
Example



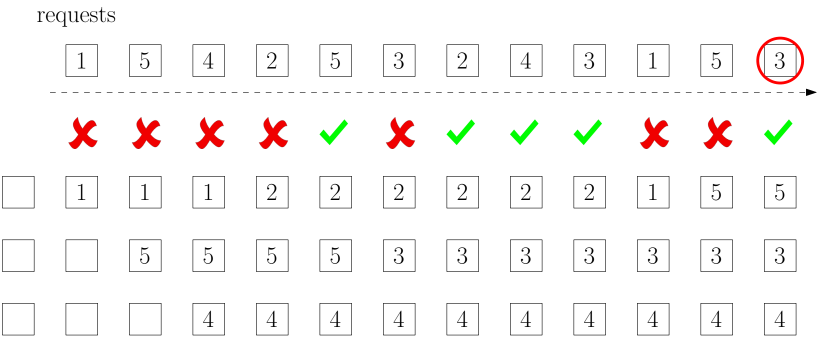
Example



Example


requests
1 5 4 2 5 3 2 4 3 1 5 3
1 1 1 2 2 2 2 2 2 1 5
5 5 5 5 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4

Example


requests
1 5 4 2 5 3 2 4 3 1 5 3
1 1 1 2 2 2 2 2 2 1 5 5
5 5 5 5 3 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4 4

Recall: Designing and Analyzing Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is "safe" (key)
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

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Offline Caching Problem

Input: k : the size of cache

n : number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_t \in \{\mathrm{hit}, \text{empty}\} \cup [n]$

- empty stands for an empty page
- "hit" means evicting no pages

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Input: k : the size of cache

n : number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

$p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

Output: $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit}, \text{empty}\} \cup [n]$

- empty stands for an empty page
- "hit" means evicting no pages