Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

```
F ← ∅
𝒮 ← {{v} : v ∈ V}
sort the edges of E in non-decreasing order of weights w
for each edge (u, v) ∈ E in the order do
    S_u ← the set in 𝒮 containing u
    S_v ← the set in 𝒮 containing v
    if S_u ≠ S_v then
        F ← F ∪ {(u, v)}
        𝒮 ← 𝒮 \ {S_u} \ {S_v} ∪ {S_u ∪ S_v}
return (V, F)
```

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

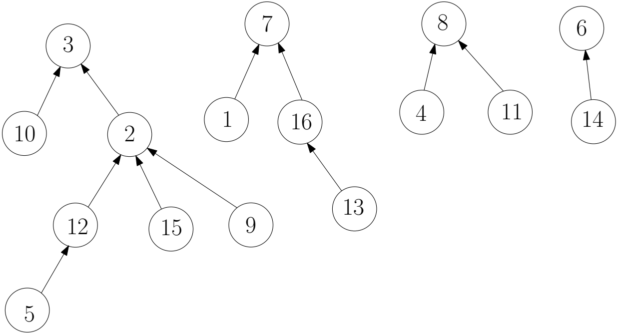
1: $F \leftarrow \emptyset$
2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of E in non-decreasing order of weights w
4: **for** each edge $(u, v) \in E$ in the order **do**
5: $\quad S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6: $\quad S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7: $\quad$**if** $S_u \neq S_v$ **then**
8: $\quad\quad F \leftarrow F \cup \{(u, v)\}$
9: $\quad\quad \mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: **return** (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

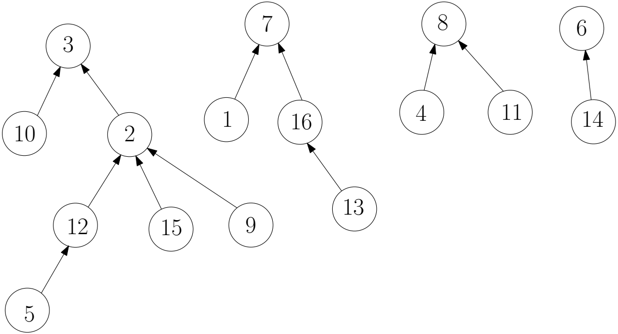
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] = \perp$ if i is a root).

Union-Find Data Structure



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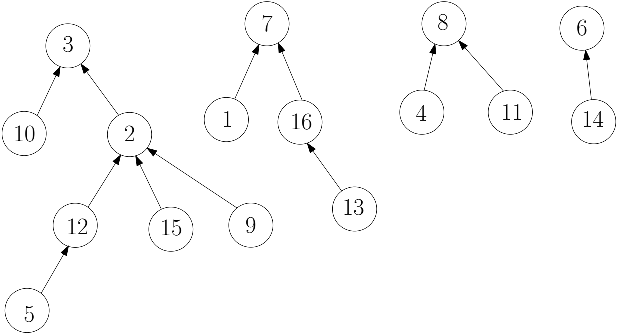

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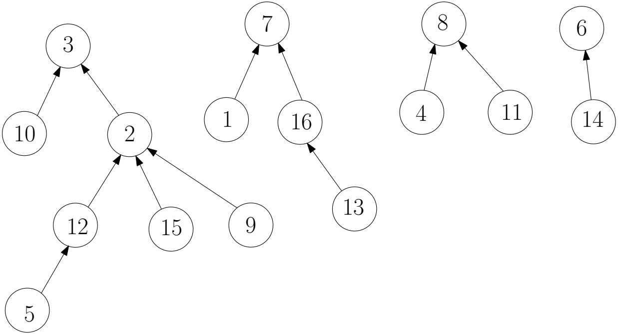

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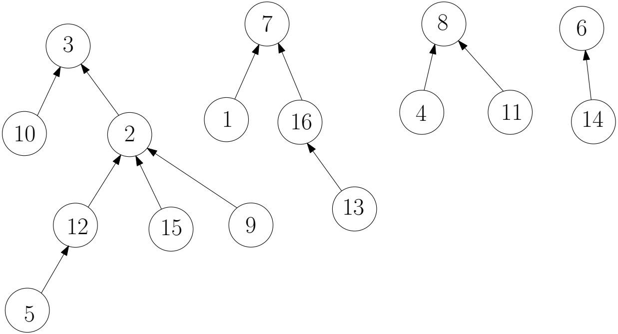

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- root(u): the root of the tree containing u

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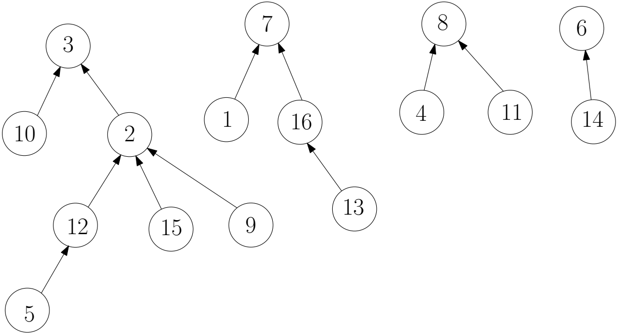

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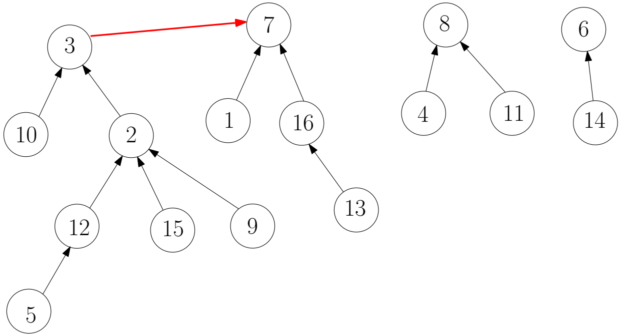

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Union-Find Data Structure

root(v)

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if par[v] = ⊥ then
    return v
else
    return root(par[v])
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Union-Find Data Structure

root(v)

1: **if** $par[v] = \perp$ **then**
2:     **return** v
3: **else**
4:     **return** root($par[v]$)

- Problem: the tree might too deep; running time might be large

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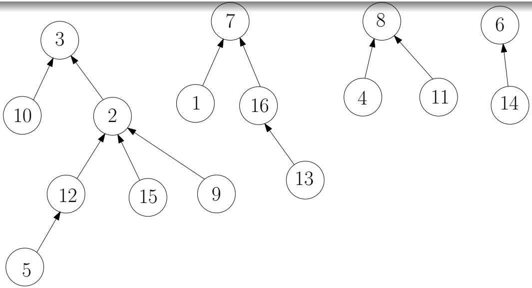

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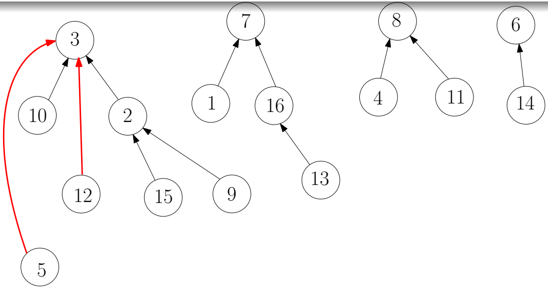

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for each edge $(u, v) \in E$ in the order **do**
    $u' \leftarrow \text{root}(u)$
    $v' \leftarrow \text{root}(v)$
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        $F \leftarrow F \cup \{(u, v)\}$
        $par[u'] \leftarrow v'$
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- 2, 5, 6, 7, 9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \le 4$ for $n \le 10^{80}$.

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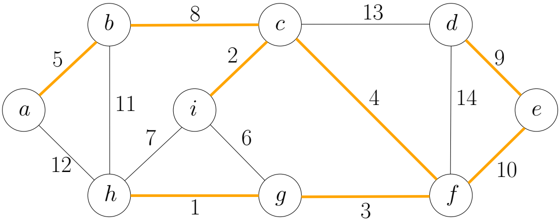
- 2, 5, 6, 7, 9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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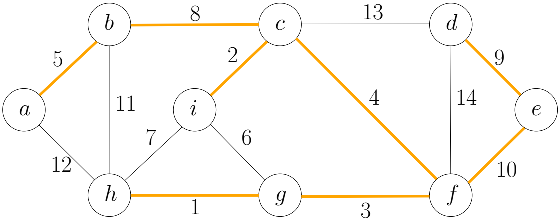

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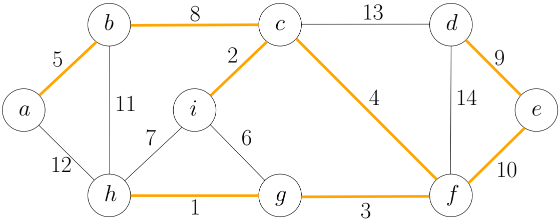

- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree

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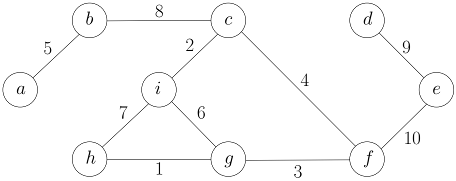

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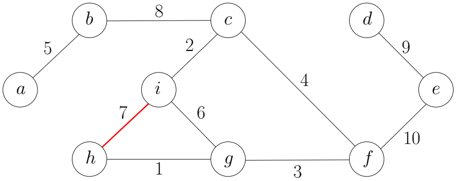

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A: The heaviest non-bridge edge.

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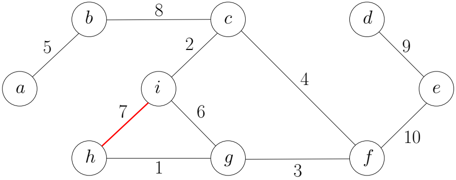


Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

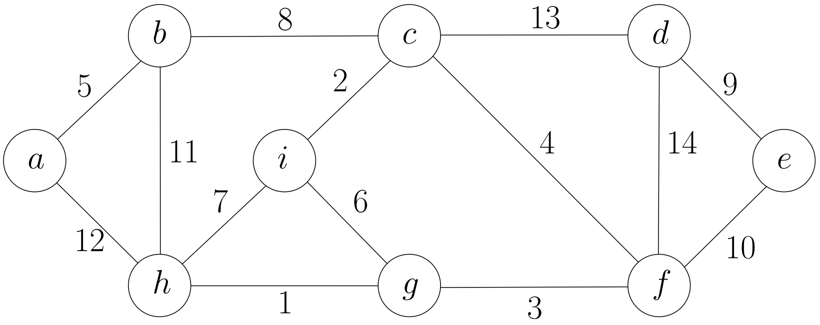
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

$F \leftarrow E$
sort E in non-increasing order of weights
for every e in this order **do**
    if $(V, F \setminus \{e\})$ is connected **then**
        $F \leftarrow F \setminus \{e\}$
return (V, F)

Reverse Kruskal's Algorithm: Example



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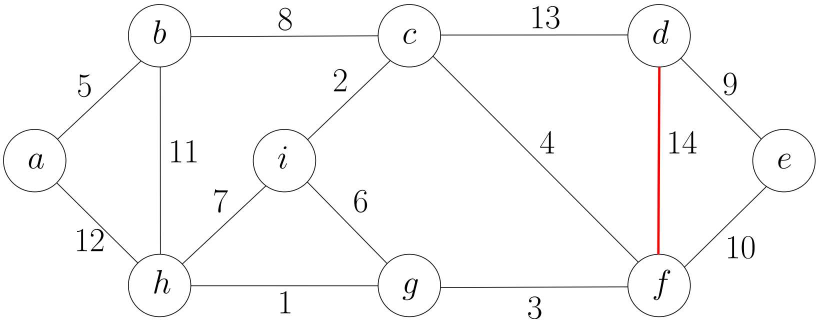

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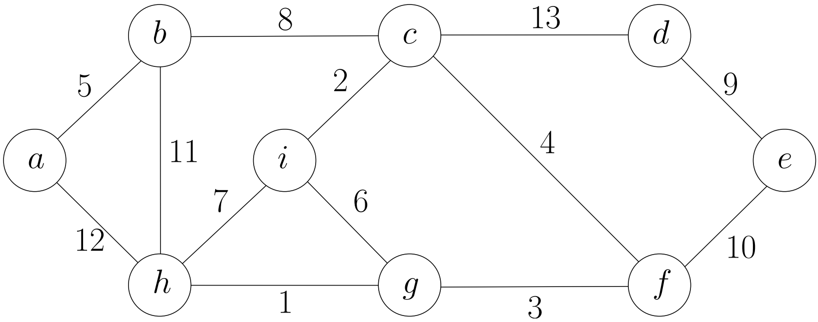

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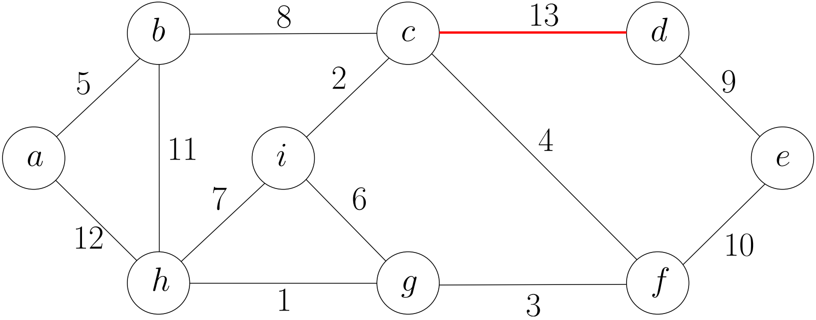

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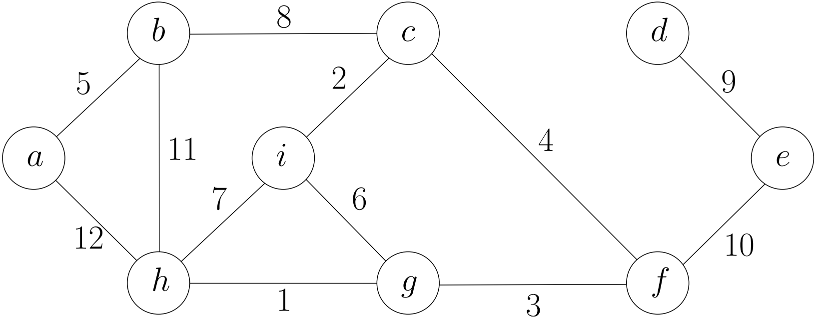

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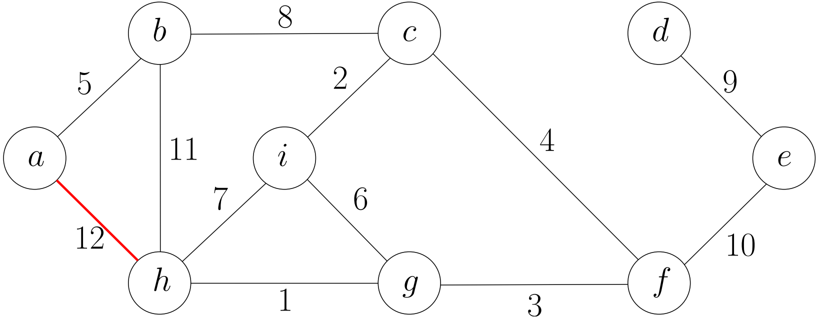

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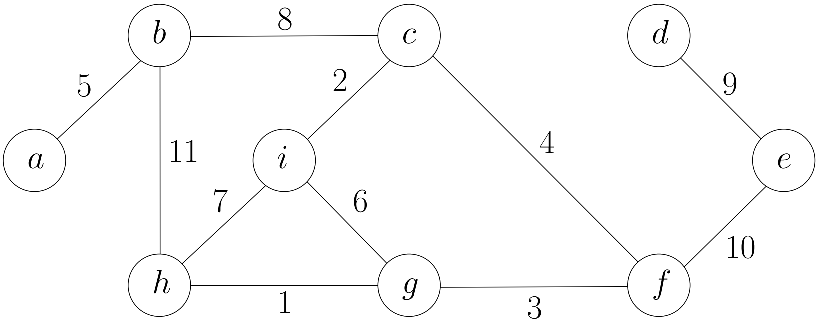

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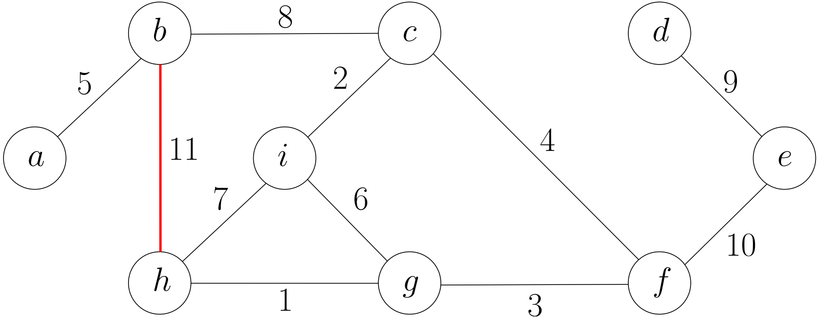

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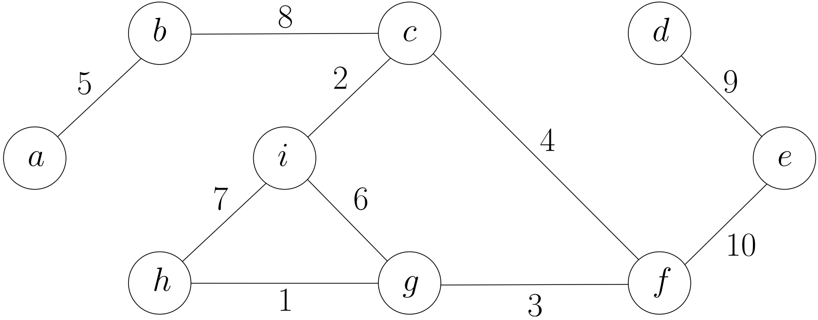

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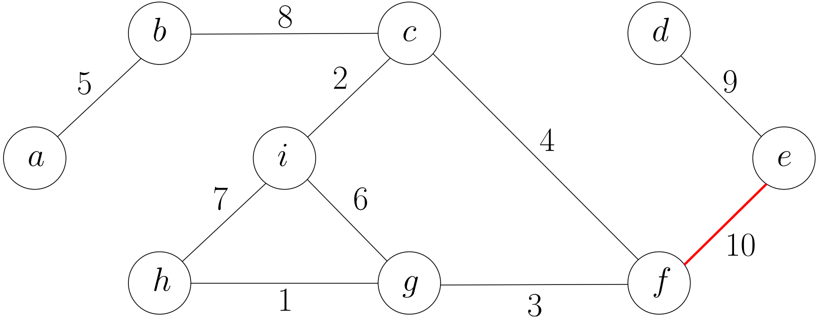

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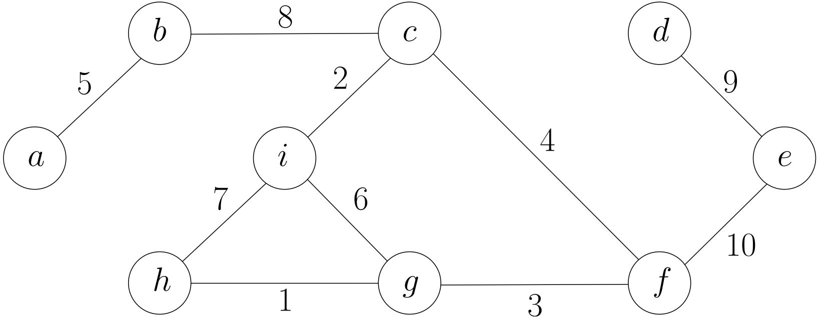

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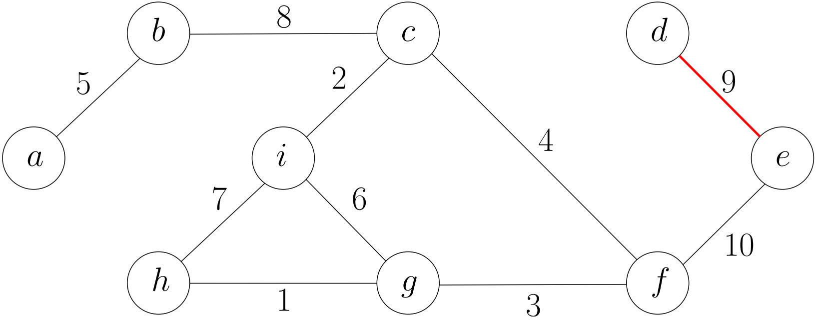

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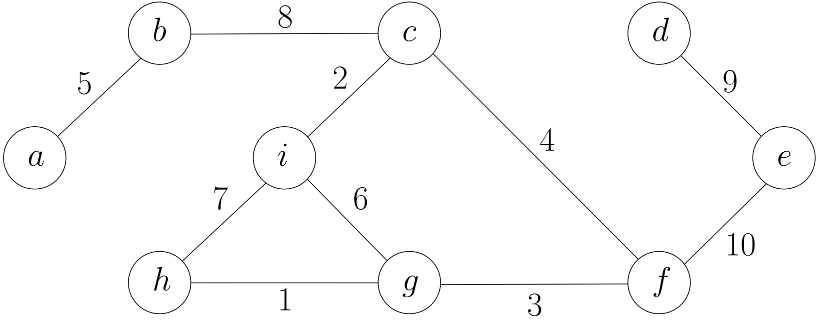

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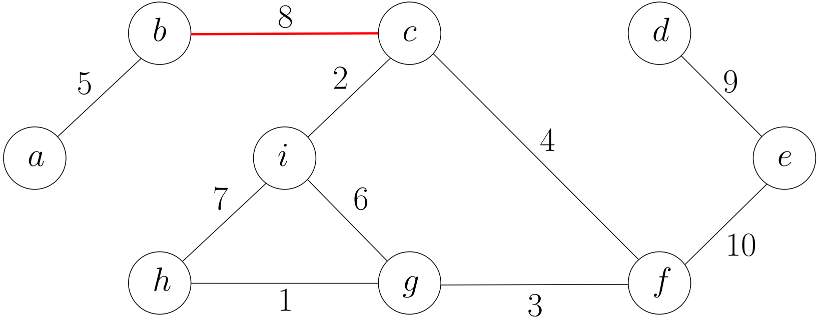

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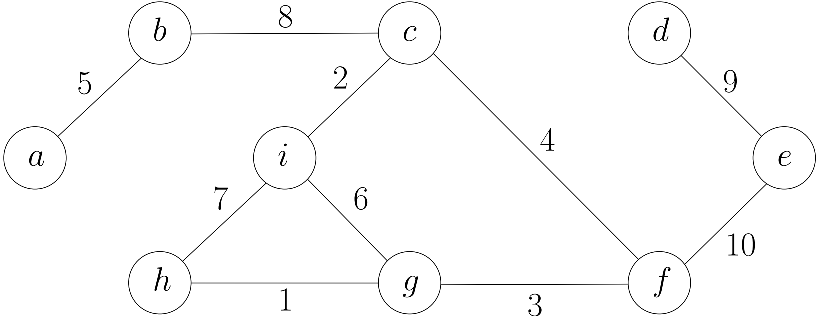

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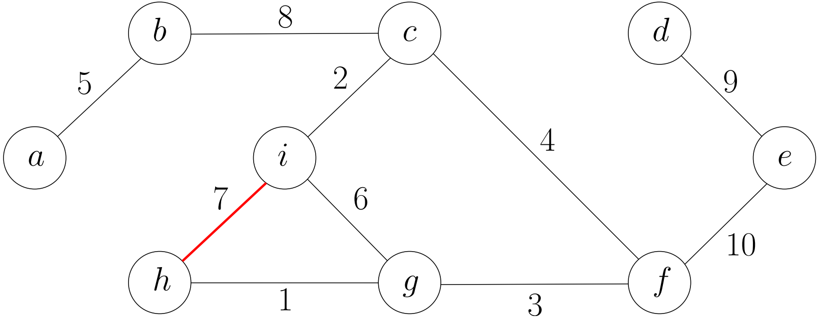

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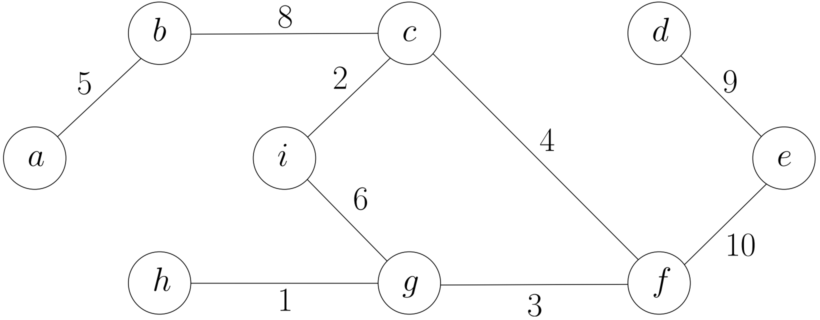

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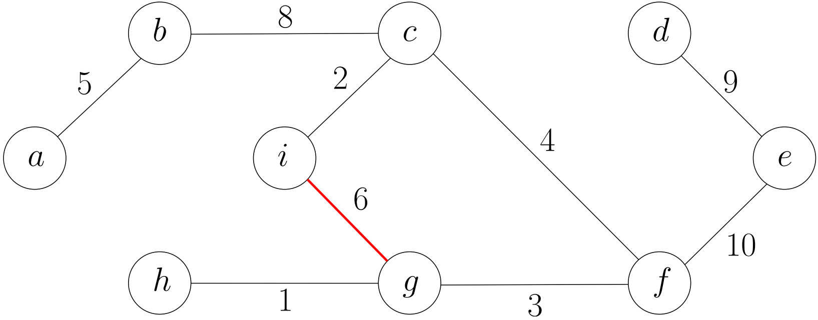

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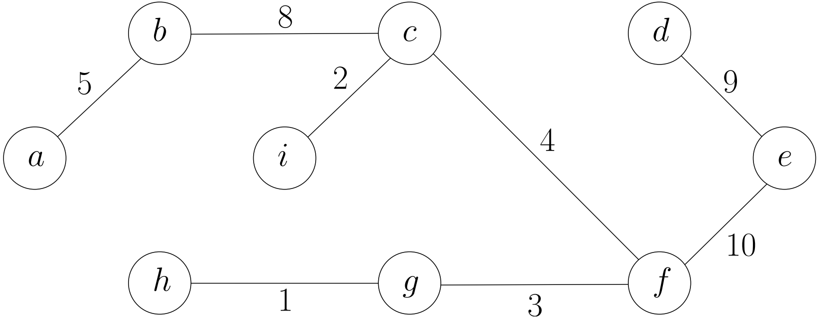

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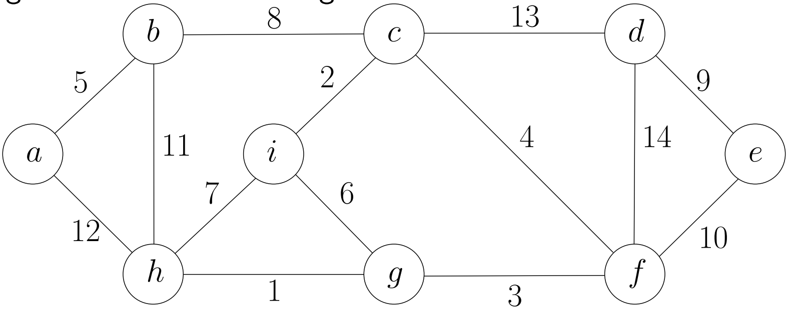
- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



Design Greedy Strategy for MST

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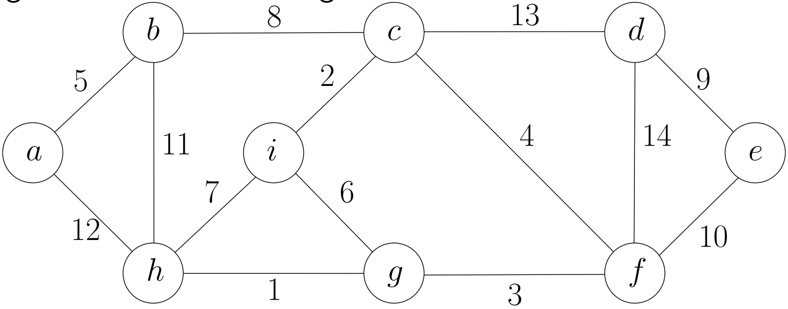

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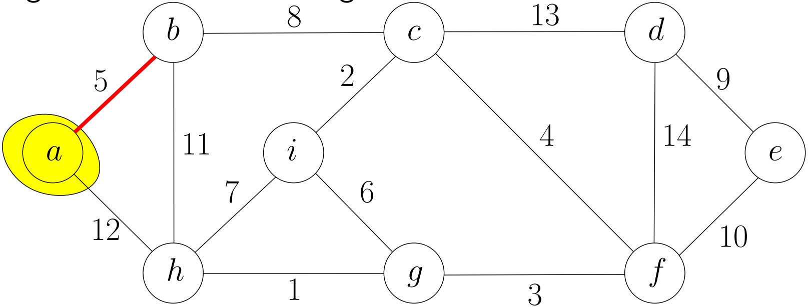

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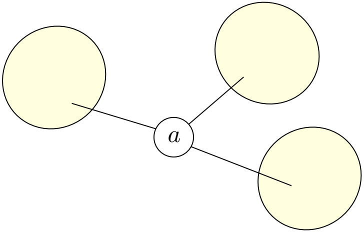

Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

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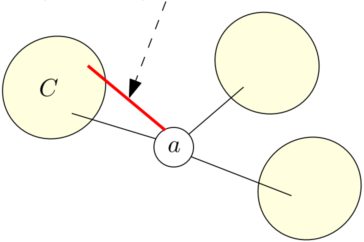

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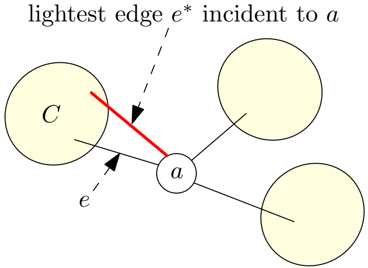

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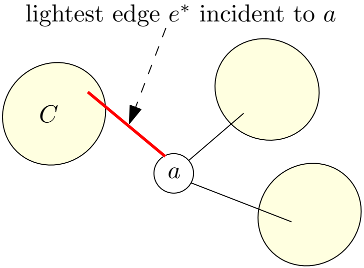

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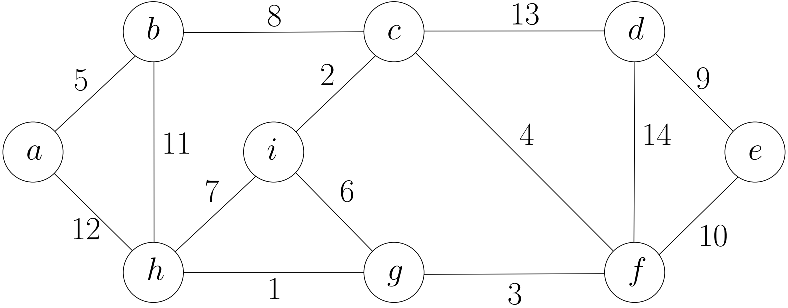
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ $\square$

Prim's Algorithm: Example



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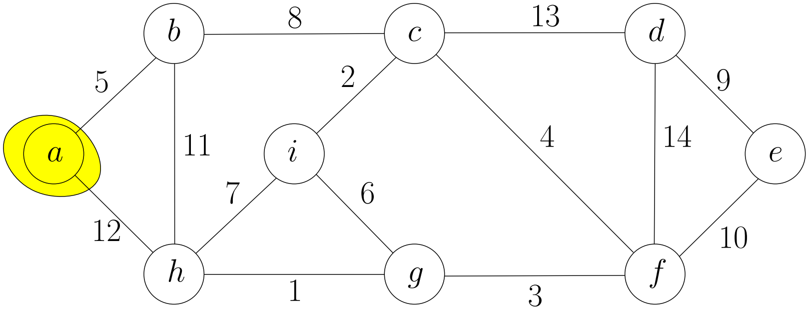
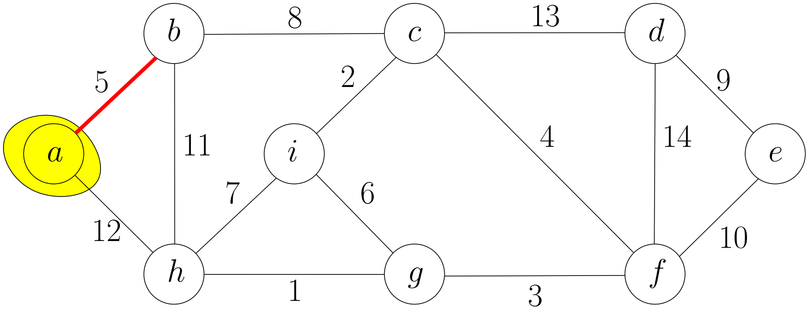
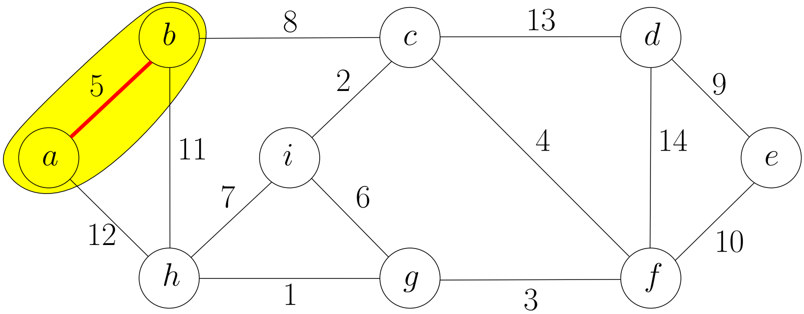


Prim's Algorithm: Example



Prim's Algorithm: Example



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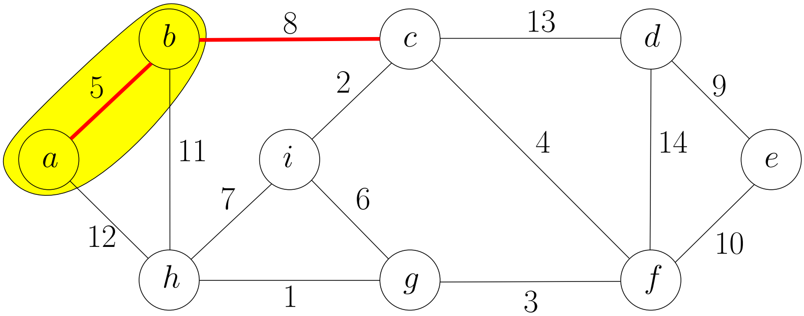

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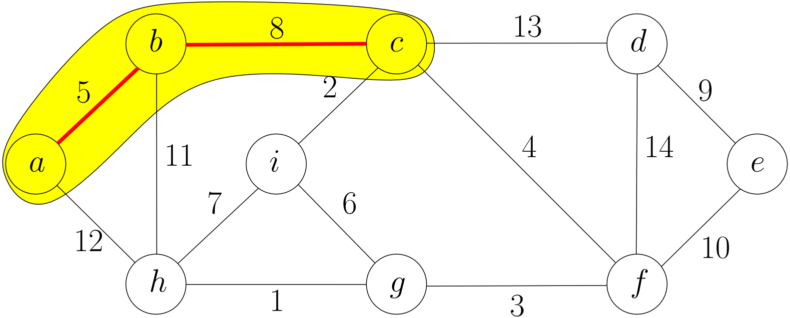

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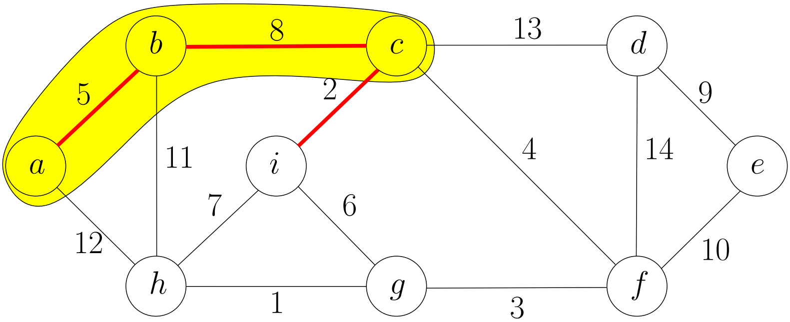

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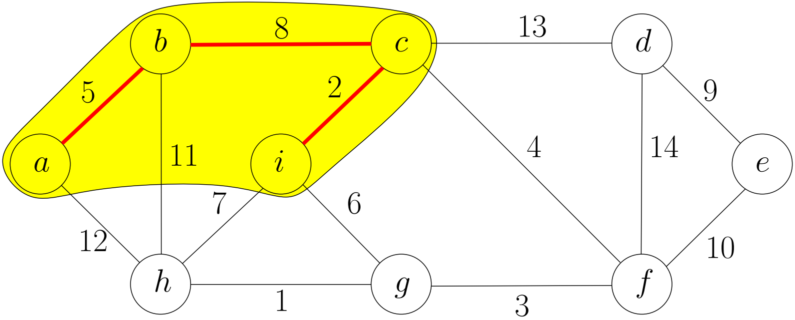

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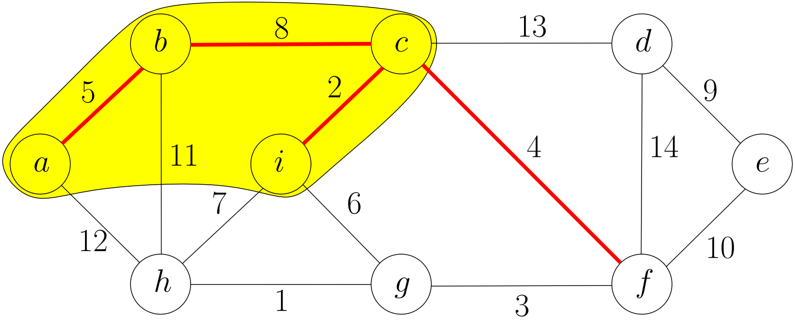

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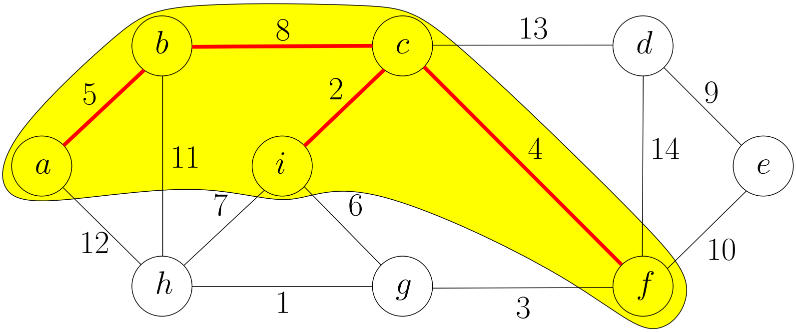

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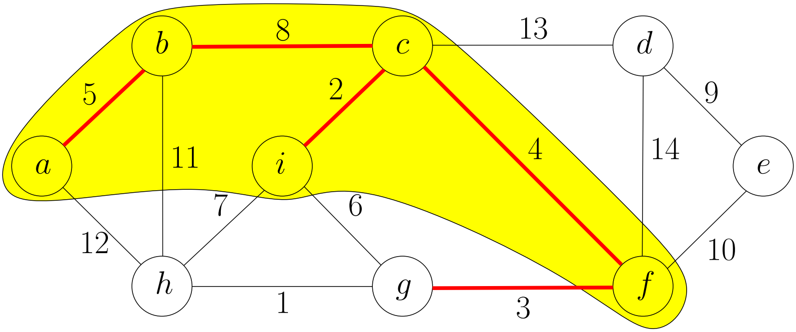

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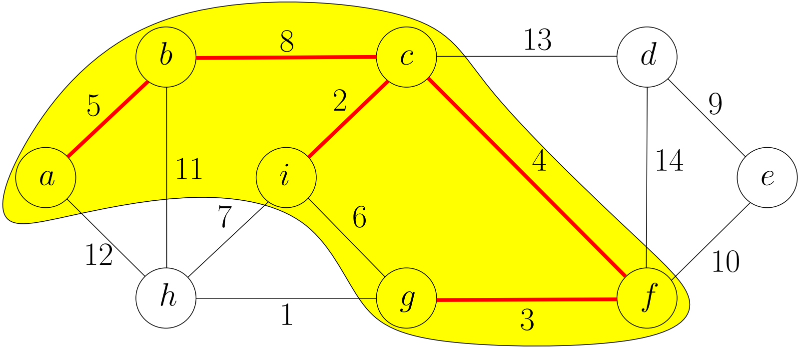

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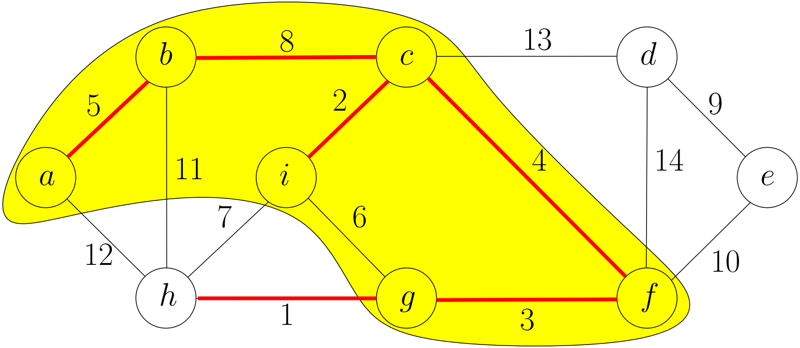

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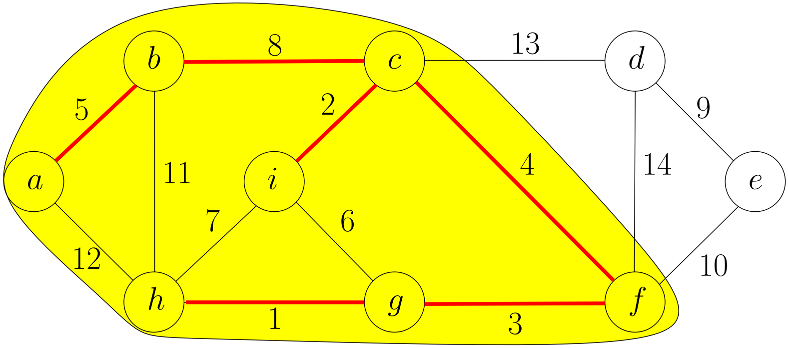

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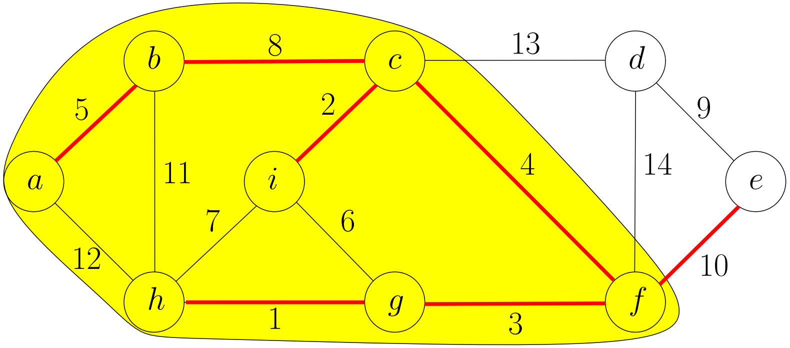

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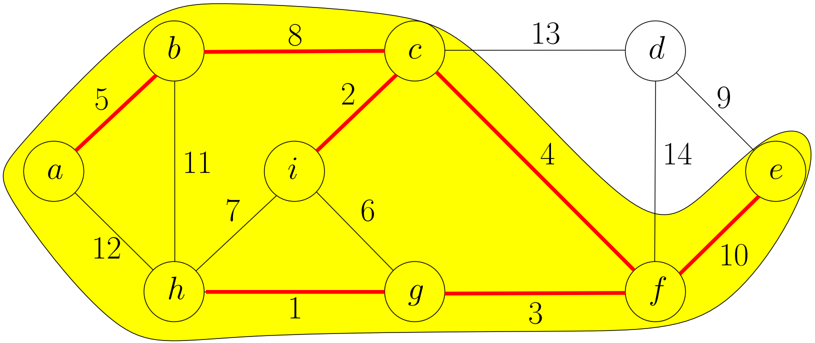

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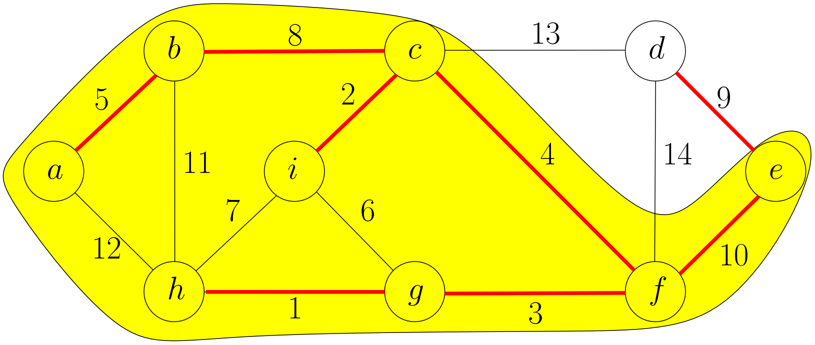

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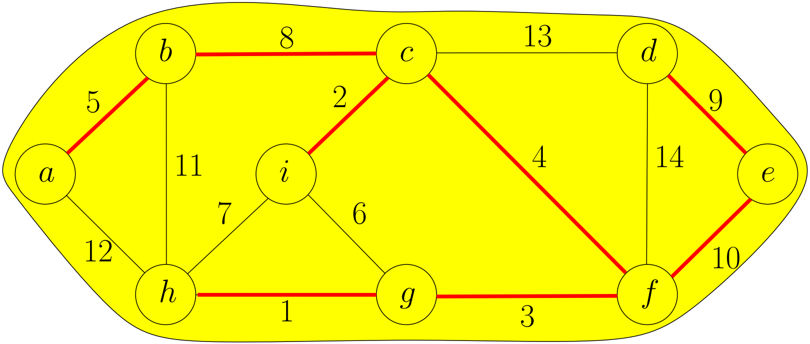


Greedy Algorithm

MST-Greedy1(G, w)

$S \leftarrow \{s\}$, where s is arbitrary vertex in V
$F \leftarrow \emptyset$
while $S \neq V$ **do**
    $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
        where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

Greedy Algorithm

MST-Greedy1(G, w)

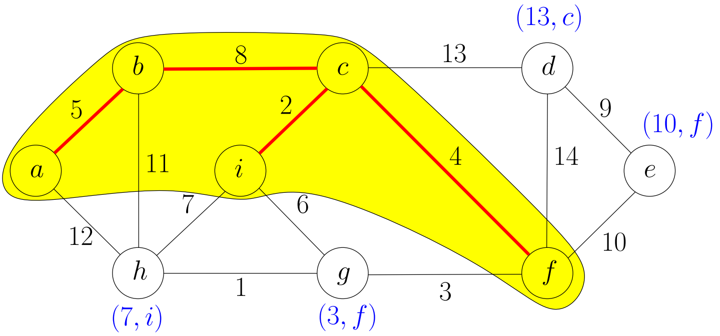
```
S ← {s}, where s is arbitrary vertex in V
F ← ∅
while S ≠ V do
    (u, v) ← lightest edge between S and V \ S,
                              where u ∈ S and v ∈ V \ S
    S ← S ∪ {v}
    F ← F ∪ {(u, v)}
return (V, F)
```

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S



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In every iteration

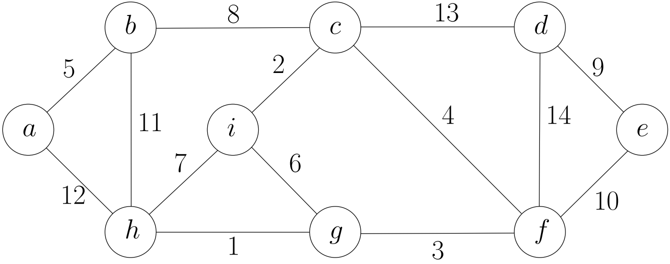
- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$
            $\pi[v] \leftarrow u$
return $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$

Example



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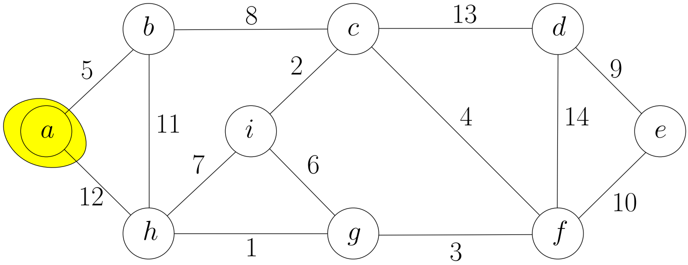
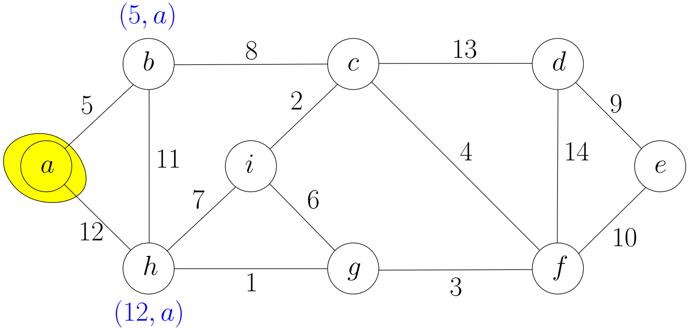


Example



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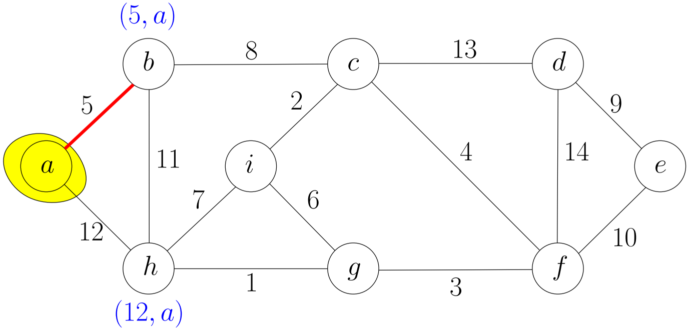

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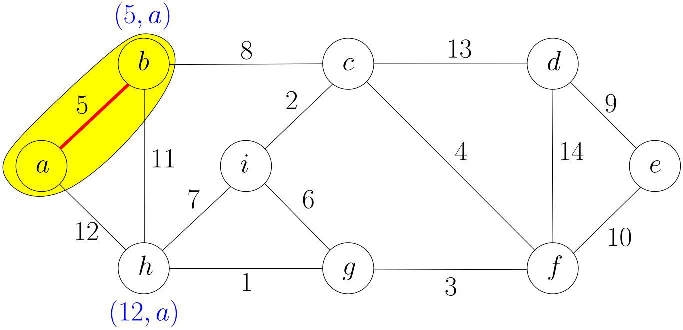

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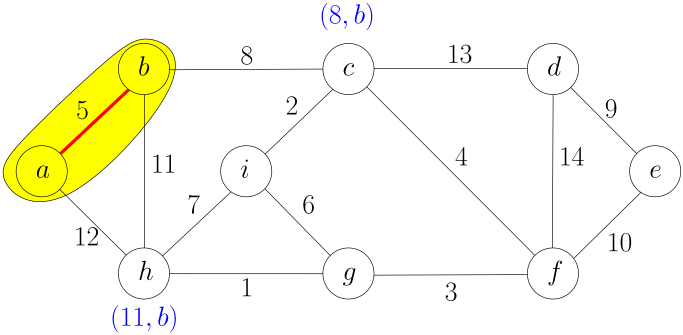

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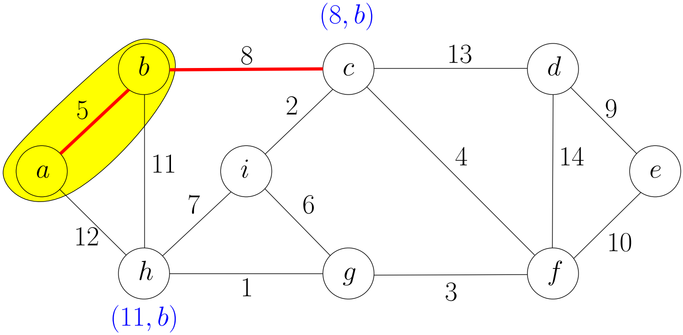

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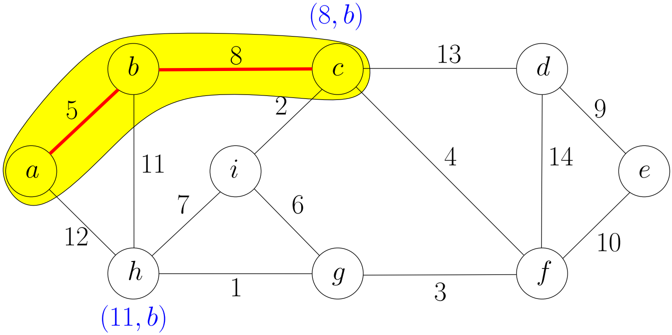

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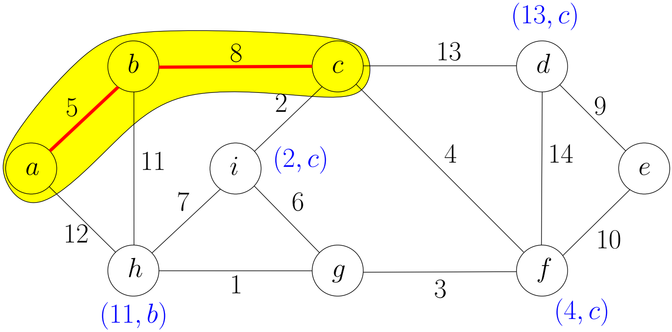

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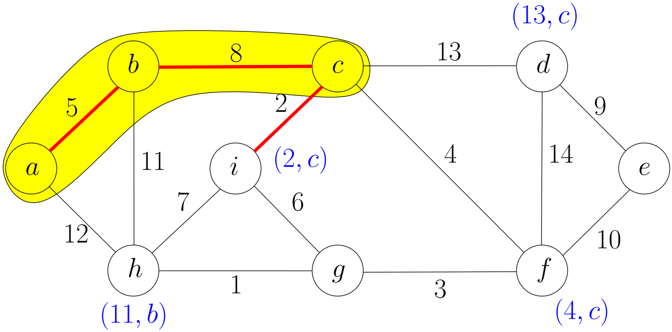

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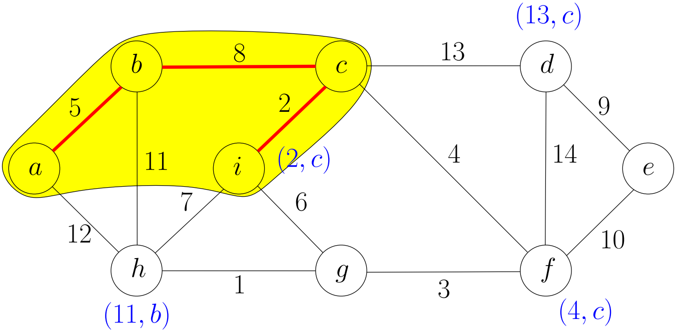

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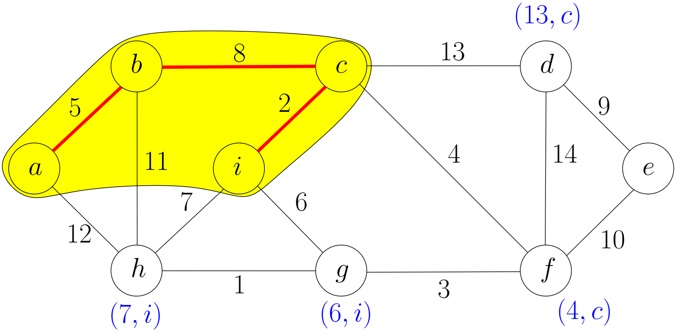

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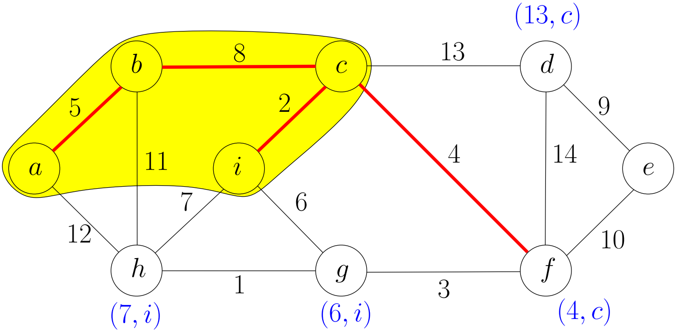

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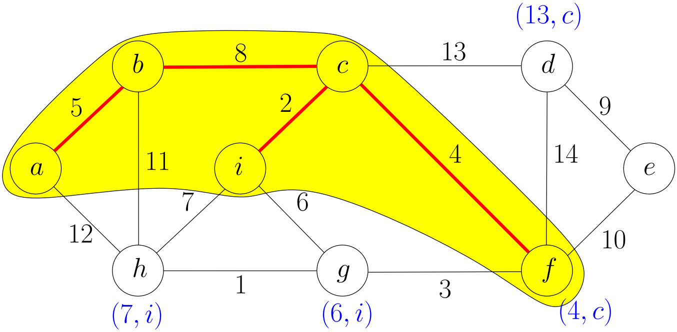

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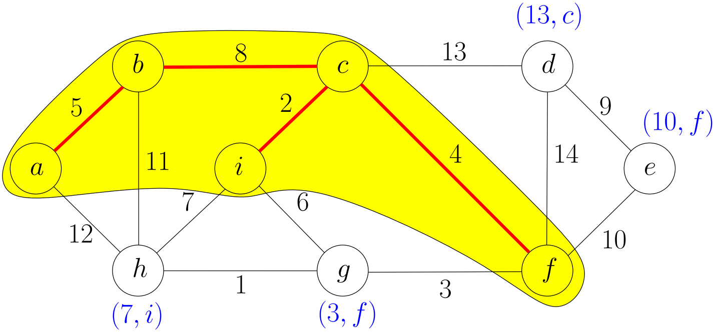

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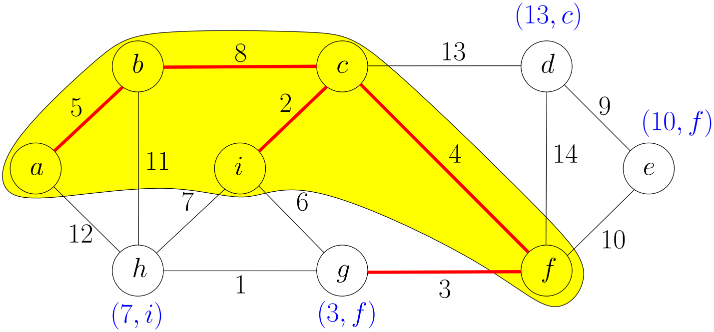

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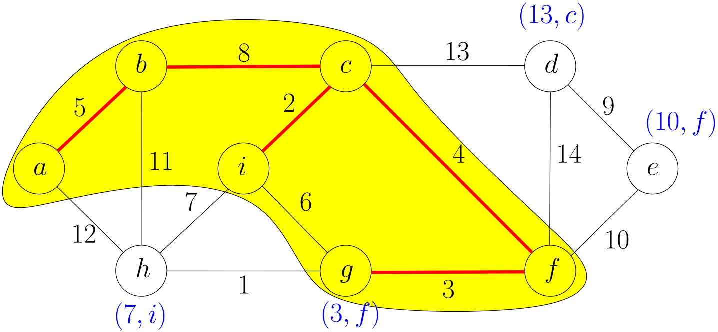

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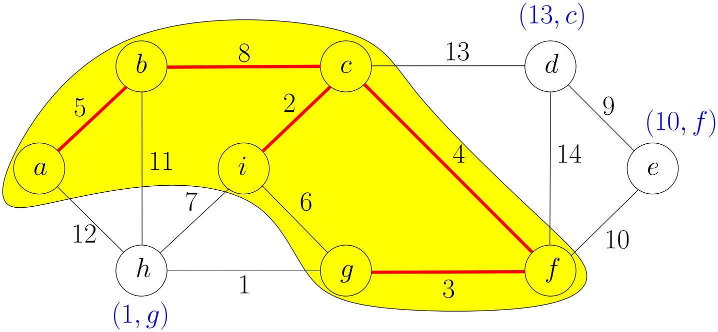

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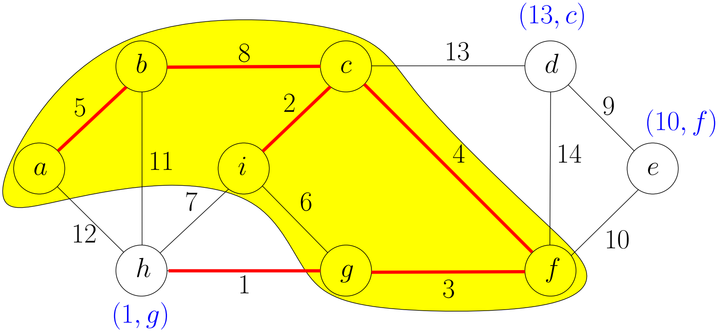

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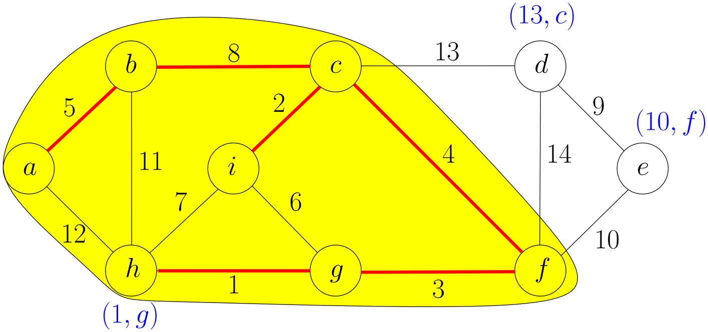

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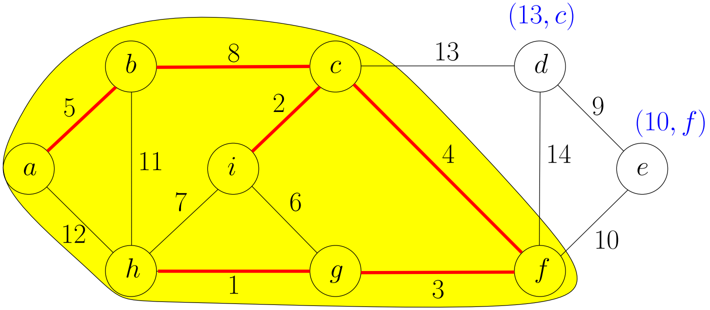

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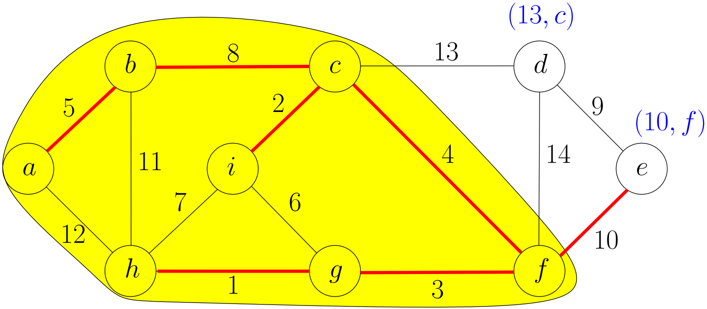

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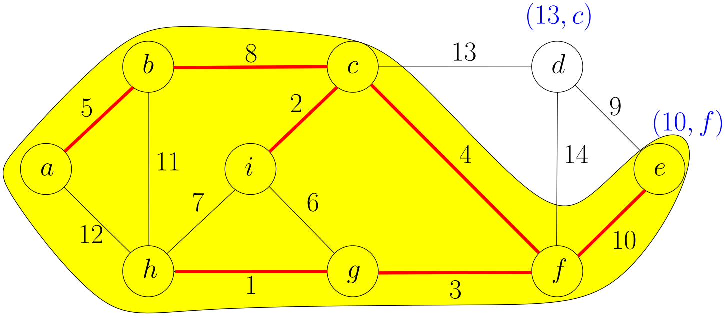

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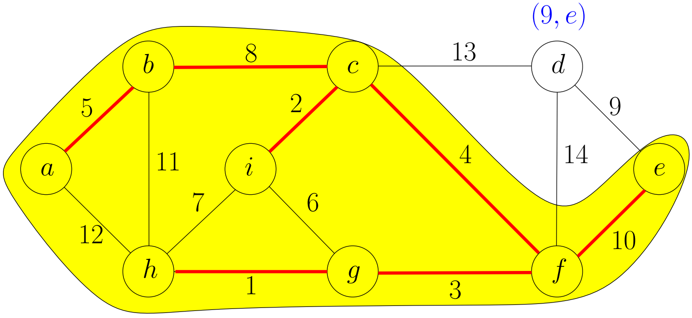

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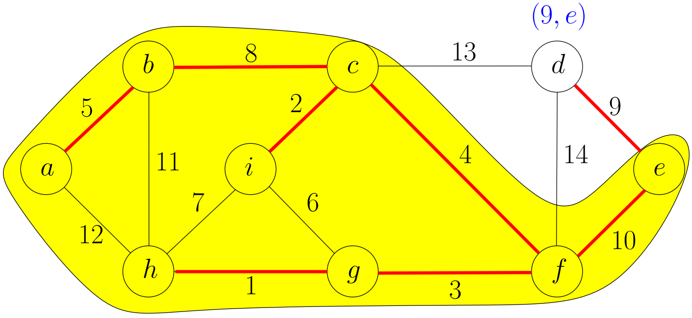

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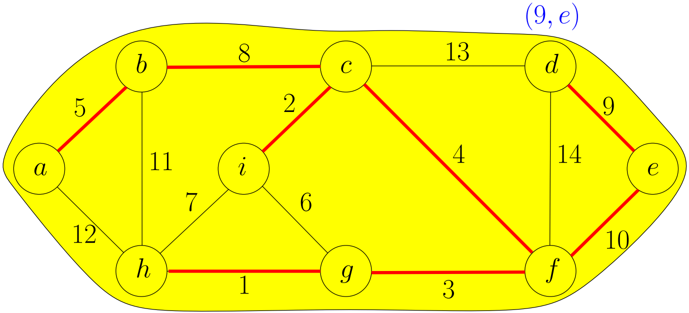

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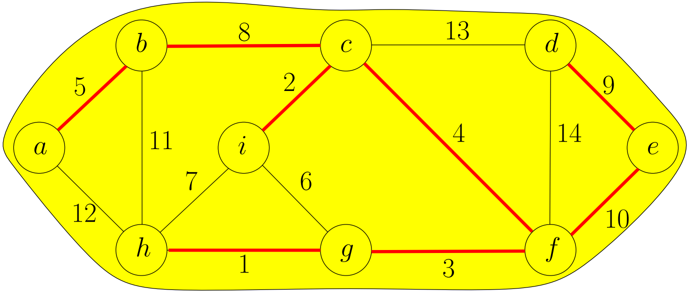

Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u, v)$:
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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value — extract_min
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$

while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$
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return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+ O(m)\times$ (time for decrease_key)

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+ O(m)\times$ (time for decrease_key)

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

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Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.

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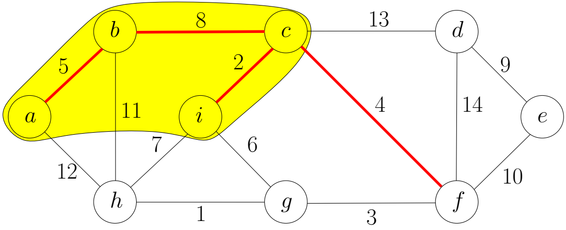

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$

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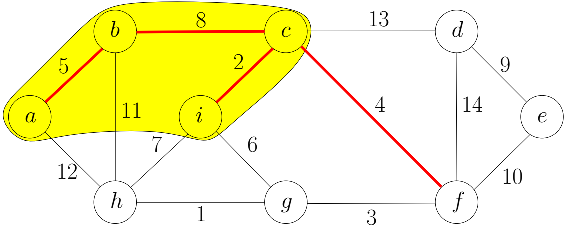

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

"Evidence" for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

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Exactly one of the following is true:

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Thus, the minimum spanning tree is unique with assumption.