Outline

1. Weighted Interval Scheduling
2. Subset Sum Problem
3. Knapsack Problem
4. Longest Common Subsequence
   - Longest Common Subsequence in Linear Space
5. Shortest Paths in Directed Acyclic Graphs
6. Matrix Chain Multiplication
7. Optimum Binary Search Tree
8. Summary
9. Summary of Studies Until April
Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse
Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.
Weighted interval scheduling: \( opt[i] = \) value of instance defined by jobs \( \{1, 2, \cdots, i\} \)

Subset sum, knapsack: \( opt[i, W'] = \) value of instance with items \( \{1, 2, \cdots, i\} \) and budget \( W' \)

Longest common subsequence: \( opt[i, j] = \) value of instance defined by \( A[1..i] \) and \( B[1..j] \)

Shortest paths in DAG: \( f[v] = \) length of shortest path from \( s \) to \( v \)

Matrix chain multiplication, optimum binary search tree: \( opt[i, j] = \) value of instances defined by matrices \( i \) to \( j \)
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  - Asymptotic analysis: $O$, $\Omega$, $\Theta$, compare the orders
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  - Polynomial time (efficient algorithm), exponential time
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- Topological Ordering problem: topological-sort algorithm
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- Greedy algorithms: safety strategy + self reduce
- Box Packing problem: greedy algorithm
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  - Exercise problems: Job scheduling with deadline, clustering problem, Coin Problem, Weighted scheduling problem
Important notations/algorithms

- **Divide-and-Conquer algorithms**: Divide + Conquer + Combine
- Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
- Counting inversions problem: sort-and-count algorithm
- Selection problem: selection algorithm based on quicksort
- Polynomial Multiplication problem: multiply algorithm
- Recurrences: recursive-tree method and Master Theorem
- Fibonacci number problem: power algorithm
- Exercise problems: Modular Exponentiation Problem, Matrix Multiplication, Closest Pair, Convex Hull
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- Edit distance with insertions and deletions problem: apply algorithm for LCS problem
- Edit distance with insertions, deletions and replacing problem
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1. Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of $G$ is a sub-graph of $G$ that is a tree including all vertices $V$. 
Lemma  Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- $T$ is a spanning tree of $G$;
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- $T$ is a spanning tree of $G$;
- $T$ is acyclic and connected;
- $T$ is minimally connected: removal of any edge disconnects it;
- $T$ is maximally acyclic: addition of any edge creates a cycle;
- $T$ has a unique simple path between every pair of nodes.
**Lemma**  Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

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How to find a spanning tree?

- BFS
How to find a spanning tree?
- BFS
- DFS
Minimum Spanning Tree (MST) Problem

**Input:** Graph \( G = (V, E) \) and edge weights \( w : E \rightarrow \mathbb{R} \)

**Output:** the spanning tree \( T \) of \( G \) with the minimum total weight
Minimum Spanning Tree (MST) Problem

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Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice
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Two Classic Greedy Algorithms for MST

- Kruskal’s Algorithm
- Prim’s Algorithm
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4 All-Pair Shortest Paths and Floyd-Warshall
Q: Which edge can be safely included in the MST?
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A: The edge with the smallest weight (lightest edge).
**Lemma** It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.
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Proof.
- Take a minimum spanning tree $T$

![Diagram of a minimum spanning tree]
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$

![Diagram of a graph with a minimum spanning tree and a lightest edge marked]
Lemma  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
**Lemma** It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**

- Take a minimum spanning tree $T$
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- Remove any edge $e$ in the path to obtain tree $T'$
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- Take a minimum spanning tree $T$
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- Remove any edge $e$ in the path to obtain tree $T'$
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: $T'$ is also a MST
Is the Residual Problem Still a MST Problem?

Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

Contract the edge \((g, h)\)

Residual problem: find the minimum spanning tree in the contracted graph
Is the Residual Problem Still a MST Problem?

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Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

**Contract** the edge \((g, h)\)

Residual problem: find the minimum spanning tree in the contracted graph
Contraction of an Edge \((u, v)\)

1. Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\).
2. Remove all edges \((u, v)\) from \(E\).
3. For every edge \((u, w)\) \(\in E\), change it to \((u^*, w)\).
4. For every edge \((v, w)\) \(\in E\), change it to \((u^*, w)\).

May create parallel edges!

E.g.: two edges \((i, g^*)\)
Contraction of an Edge \((u, v)\)

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- For every edge \((v, w)\) \(\in\) \(E\), \(w \neq u\), change it to \((u^*, w)\)
- May create parallel edges! E.g.: two edges \((i, g^*)\)
Greedy Algorithm

Repeat the following step until $G$ contains only one vertex:

1. Choose the lightest edge $e^*$, add $e^*$ to the spanning tree
2. Contract $e^*$ and update $G$ be the contracted graph
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Q: What edges are removed due to contractions?
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1. Choose the lightest edge $e^*$, add $e^*$ to the spanning tree
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Q: What edges are removed due to contractions?

A: Edge $(u, v)$ is removed if and only if there is a path connecting $u$ and $v$ formed by edges we selected
Greedy Algorithm

MST-Greedy($G, w$)

1: $F \leftarrow \emptyset$
2: sort edges in $E$ in non-decreasing order of weights $w$
3: for each edge $(u, v)$ in the order do
4: if $u$ and $v$ are not connected by a path of edges in $F$ then
5: $F \leftarrow F \cup \{(u, v)\}$
6: return $(V, F)$
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}
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Sets: \( \{a, b, c, i, f, g, h\} \), \( \{d, e\} \)
Kruskal’s Algorithm: Example

Sets: \{a, b, c, i, f, g, h, d, e\}
Kruskal’s Algorithm: Efficient Implementation of Greedy Algorithm

**MST-Kruskal**($G$, $w$)

1. $F \leftarrow \emptyset$
2. $S \leftarrow \{\{v\} : v \in V\}$
3. sort the edges of $E$ in non-decreasing order of weights $w$
4. **for** each edge $(u, v) \in E$ in the order **do**
5. \quad $S_u \leftarrow$ the set in $S$ containing $u$
6. \quad $S_v \leftarrow$ the set in $S$ containing $v$
7. \quad **if** $S_u \neq S_v$ **then**
8. \quad \quad $F \leftarrow F \cup \{(u, v)\}$
9. \quad \quad $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. **return** ($V, F$)