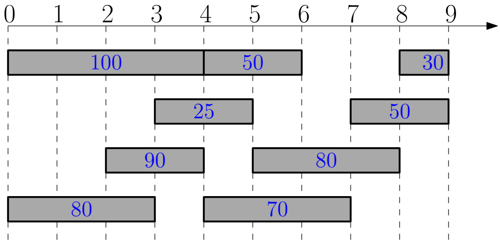
Weighted Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-weight subset of mutually compatible jobs



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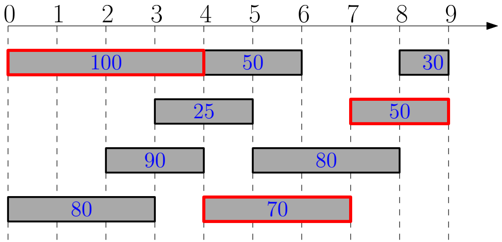

Optimum value $= 220$

Hard to Design a Greedy Algorithm

Q: Which job is safe to schedule?

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- Job with the earliest finish time?

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- Job with the largest weight?

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  No, when weights are equal, this is the shortest job

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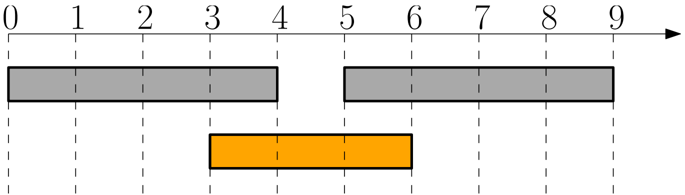
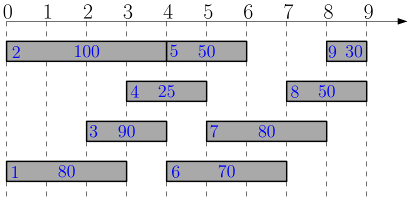


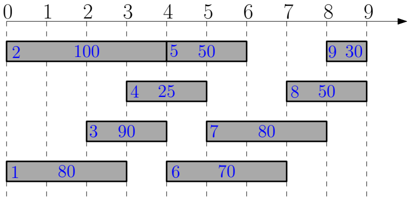
Designing a Dynamic Programming Algorithm

Designing a Dynamic Programming Algorithm



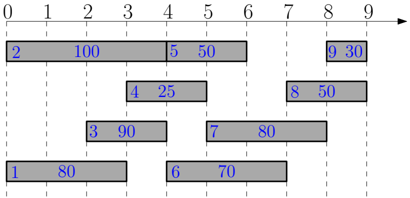
- Sort jobs according to non-decreasing order of finish times

Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

- Sort jobs according to non-decreasing order of finish times
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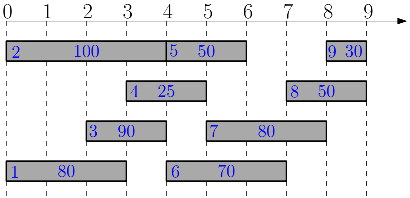
Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	0
1	
2	
3	
4	
5	
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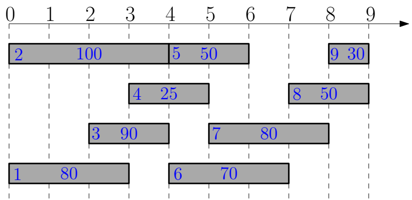

i	$opt[i]$
0	0
1	80
2	
3	
4	
5	
6	
7	
8	
9	

- Sort jobs according to non-decreasing order of finish times
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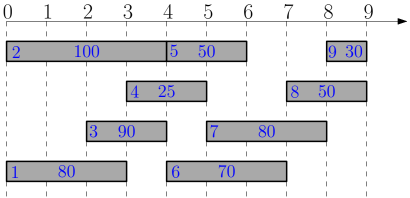
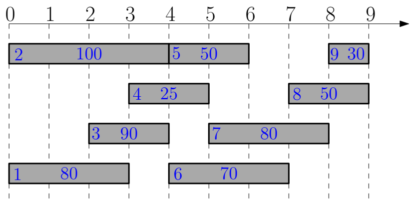
Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

i	$opt[i]$
0	0
1	80
2	100
3	
4	
5	
6	
7	
8	
9	

Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	0
1	80
2	100
3	100
4	
5	
6	
7	
8	
9	

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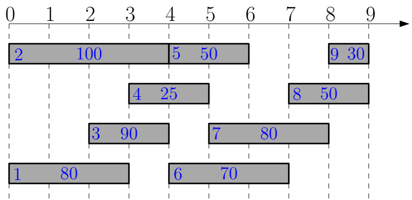

i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

- Sort jobs according to non-decreasing order of finish times
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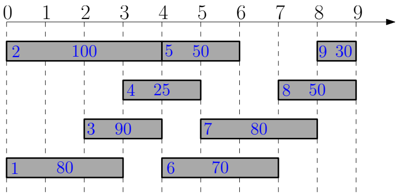
Designing a Dynamic Programming Algorithm



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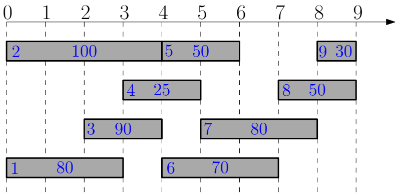
- $opt[i]$: optimal value for the instance

Designing a Dynamic Programming Algorithm



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- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

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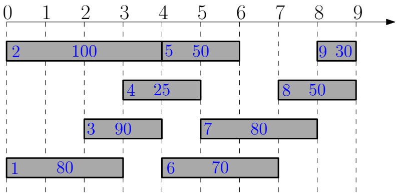

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Q: The value of optimal solution that does not contain i?

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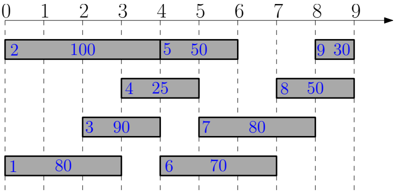

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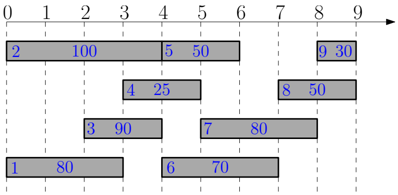

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Q: The value of optimal solution that contains job i?

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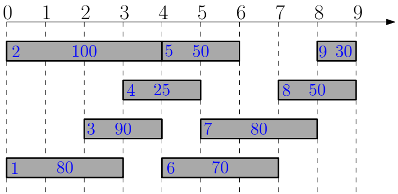

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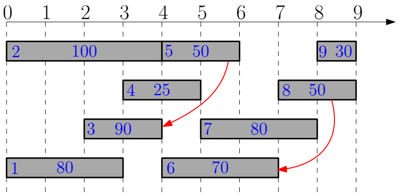
A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm



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Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

Designing a Dynamic Programming Algorithm

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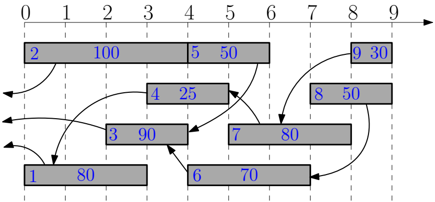

- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] =$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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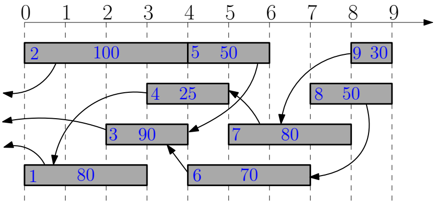

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Designing a Dynamic Programming Algorithm

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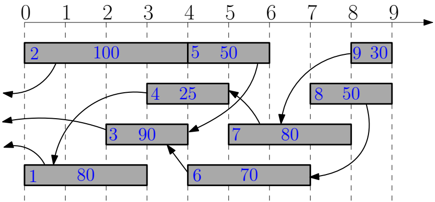

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- $opt[2] = \max\{opt[1], 100 + opt[0]\}$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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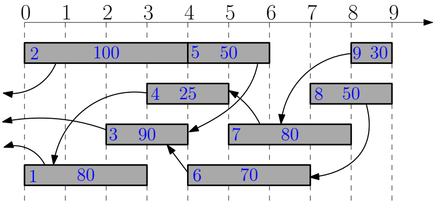

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Designing a Dynamic Programming Algorithm

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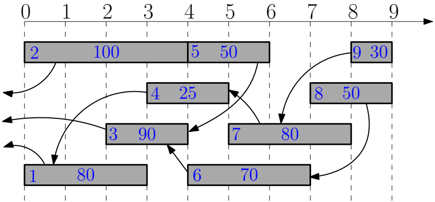

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- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\}$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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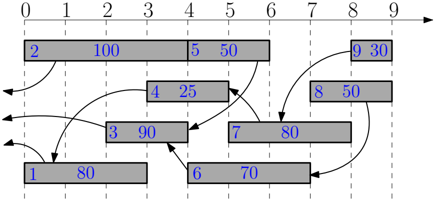

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Designing a Dynamic Programming Algorithm

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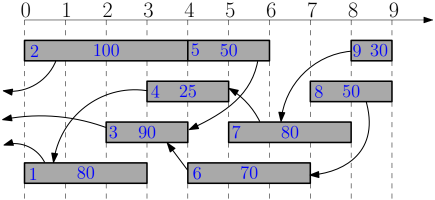

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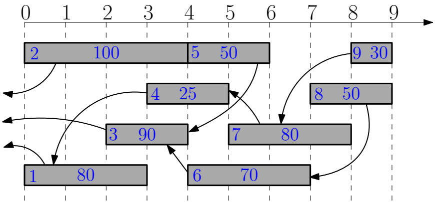
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\}$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



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- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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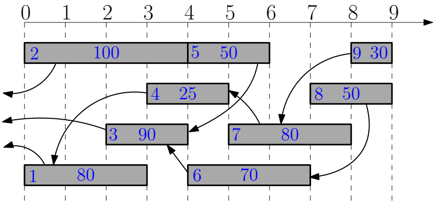

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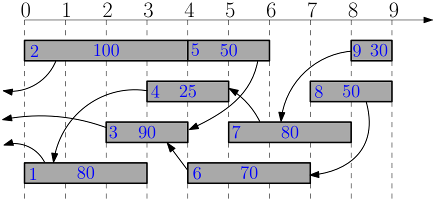
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\}$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



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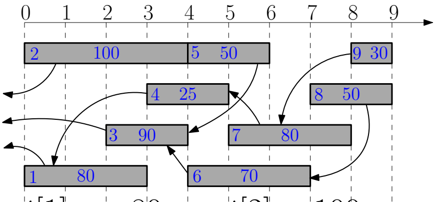
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

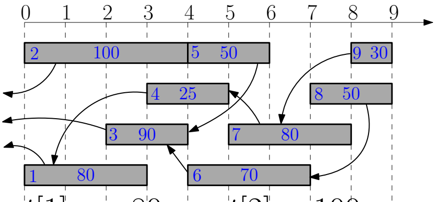


- $opt[0] = 0$, $opt[1] = 80$, $opt[2] = 100$
- $opt[3] = 100$, $opt[4] = 105$, $opt[5] = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



- $opt[0] = 0, \quad opt[1] = 80, \quad opt[2] = 100$
- $opt[3] = 100, \quad opt[4] = 105, \quad opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Dynamic Programming

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$

Dynamic Programming

```
sort jobs by non-decreasing order of finishing times
compute p_1, p_2, ..., p_n
opt[0] ← 0
for i ← 1 to n do
    opt[i] ← max{opt[i − 1], v_i + opt[p_i]}
```

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times

compute $p_1, p_2, \cdots, p_n$

$opt[0] \leftarrow 0$

for $i \leftarrow 1$ to n **do**

  if $opt[i-1] \geq v_i + opt[p_i]$ **then**

    $opt[i] \leftarrow opt[i-1]$

  else

    $opt[i] \leftarrow v_i + opt[p_i]$

How Can We Recover the Optimum Schedule?

```
sort jobs by non-decreasing order of finishing times
compute p_1, p_2, ..., p_n
opt[0] ← 0
for i ← 1 to n do
    if opt[i − 1] ≥ v_i + opt[p_i] then
        opt[i] ← opt[i − 1]
        b[i] ← N
    else
        opt[i] ← v_i + opt[p_i]
        b[i] ← Y
```

How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    if $opt[i-1] \geq v_i + opt[p_i]$ **then**
        $opt[i] \leftarrow opt[i-1]$
        $b[i] \leftarrow$ N
    else
        $opt[i] \leftarrow v_i + opt[p_i]$
        $b[i] \leftarrow$ Y

$i \leftarrow n, S \leftarrow \emptyset$
while $i \neq 0$ **do**
    if $b[i] =$ N **then**
        $i \leftarrow i - 1$
    else
        $S \leftarrow S \cup \{i\}$
        $i \leftarrow p_i$
return S