Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.

<table>
<thead>
<tr>
<th>page sequence</th>
<th>cache</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>×</td>
</tr>
<tr>
<td>5</td>
<td>×</td>
</tr>
<tr>
<td>4</td>
<td>×</td>
</tr>
<tr>
<td>2</td>
<td>×</td>
</tr>
<tr>
<td>5</td>
<td>×</td>
</tr>
<tr>
<td>3</td>
<td>×</td>
</tr>
</tbody>
</table>

misses = 6
A Better Solution for Example

<table>
<thead>
<tr>
<th>page sequence</th>
<th>cache</th>
<th></th>
<th>cache</th>
<th>misses = 6</th>
<th>misses = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗</td>
<td>1</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
<td>1 5</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>✗</td>
<td>1 5 4</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>✗</td>
<td>1 2 4</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
<td>1 2 5</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>✗</td>
<td>1 2 3</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>✗</td>
<td>1 2 3</td>
<td>✗</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>✓</td>
<td>1 2 3</td>
<td>✓</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

misses = 6
misses = 5
Offline Caching Problem

**Input:** $k$: the size of cache $n$: number of pages

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

**Output:** $i_1, i_2, i_3, \cdots, i_T \in \{\text{hit, empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)
Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

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- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.
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**Input:** \( k \) : the size of cache
\( n \) : number of pages

\( \rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n] \): sequence of requests

**Output:** \( i_1, i_2, i_3, \cdots, i_T \in \{ \text{hit}, \text{empty} \} \cup [n] \): indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

- Offline Caching: we know the whole sequence ahead of time.
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**Q:** Which one is more realistic?
### Offline Caching Problem

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**Q:** Which one is more realistic?

**A:** Online caching
- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?
- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?

A: Use the offline solution as a benchmark to measure the “competitive ratio” of online algorithms
FIFO (First-In-First-Out): Evict the first-in page in cache
FIFO (First-In-First-Out): Evict the first-in page in cache

LRU (Least-Recently-Used): Evict page whose most recent access was earliest
Offline Caching: Potential Greedy Algorithms

- **FIFO (First-In-First-Out)**: Evict the first-in page in cache
- **LRU (Least-Recently-Used)**: Evict page whose most recent access was earliest
- **LFU (Least-Frequently-Used)**: Evict page that was least frequently requested

Indeed all the algorithms are “online”, i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
Offline Caching: Potential Greedy Algorithms

- **FIFO (First-In-First-Out):** Evict the first-in page in cache
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All the above algorithms are not optimum! Indeed all the algorithms are “online”, i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
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*All the above algorithms are not optimum!*  
Indeed all the algorithms are “online”, i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
FIFO is not optimum

requests

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td></td>
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</tbody>
</table>
FIFO is not optimum
FIFO is not optimum

requests

1
2
3
4
1

FIFO

[Diagram of FIFO requests with the first request marked as incorrect]
FIFO is not optimum

Requests:

1
2
3
4
1

FIFO:

[Diagram showing the FIFO process with requests and arrows indicating movement]
FIFO is not optimum
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>✓</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
FIFO is not optimum

requests

1
2
3
4
1

FIFO

<p>| | | |</p>
<table>
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× 1 1 2 3
FIFO is not optimum

requests

1
2
3
4

FIFO

1
2
3
4
FIFO is not optimum

requests

1  x
2  x
3  x
4  x

FIFO

1
2
3
4
FIFO is not optimum

requests

1
2
3
4
1

FIFO

1
2
3
4
2
3
4
2
3
FIFO is not optimum
FIFO is not optimum

<table>
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<th>FIFO</th>
<th>misses = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>🆕</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>🆕</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>🆕</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>🆕</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>✗</td>
<td></td>
</tr>
</tbody>
</table>
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
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<td>2</td>
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<td>4</td>
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<tr>
<td>1</td>
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</tbody>
</table>

FIFO:
- Misses = 5

Furthest-in-Future:
- Misses = 4
Optimum Offline Caching

Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.
Furthest-in-Future (FF)

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>🗑️ 1 2 3</td>
<td>🗑️ 1 2 3</td>
</tr>
<tr>
<td>2</td>
<td>🗑️ 1 2 3</td>
<td>🗑️ 1 2 3</td>
</tr>
<tr>
<td>3</td>
<td>🗑️ 1 2 3</td>
<td>🗑️ 1 2 3</td>
</tr>
<tr>
<td>4</td>
<td>🗑️ 4 2 3</td>
<td>🗑️ 1 4 3</td>
</tr>
<tr>
<td>1</td>
<td>🗑️ 4 1 3</td>
<td>✓ 1 4 3</td>
</tr>
</tbody>
</table>

misses = 5

misses = 4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X

1  1  1

5  5

4
Example

requests

\[
\begin{array}{cccccccc}
1 & 5 & 4 & 2 & 5 & 3 & 2 & 4 & 3 & 1 & 5 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
\times & \times & \times \\
1 & 1 & 1 \\
5 & 5 \\
4 \\
\end{array}
\]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

x  x  x  x

1  1  1  2

5  5  5

4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X  X

1  1  1  2

5  5  5

4  4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

- - - - ✓

☐ 1 1 1 2 2
☐ ☐ 5 5 5 5
☐ ☐ ☐ 4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

× × × × ✓

☐  1  1  1  2  2

☐  ☐  5  5  5  5

☐  ☐  ☐  4  4  4
Example

requests

| 1 | 5 | 4 | 2 | 5 | 3 | 2 | 4 | 3 | 1 | 5 | 3 |

- X - X - X - X - ✓ - X

| 1 | 1 | 1 | 2 | 2 | 2 |
| 5 | 5 | 5 | 5 | 5 | 3 |
| 4 | 4 | 4 | 4 | 4 |
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

❌ ❌ ❌ ❌ ✔️ ❌

☐ 1 1 1 2 2 2

☐ ☐ 5 5 5 5 3

☐ ☐ ☐ 4 4 4 4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

x x x x ✓ x ✓ ✓

☐ 1 1 1 2 2 2 2

☐ ☐ 5 5 5 5 3 3

☐ ☐ ☐ 4 4 4 4 4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

❌ ❌ ❌ ❌ ✔️ ❌ ✔️ ✔️

☐ 1 1 1 2 2 2 2 2
☐ ☐ 5 5 5 5 3 3 3
☐ ☐ ☐ 4 4 4 4 4 4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

[Diagram of requests with crosses and checks]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

x  x  x  x  ✓  x  ✓  ✓  ✓

1  1  1  2  2  2  2  2  2

5  5  5  5  3  3  3  3

4  4  4  4  4  4  4  4

3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×  ✓  ×  ✓  ✓  ✓  ×

□  □  1  1  1  2  2  2  2  2  2  1

□  □  □  5  5  5  5  3  3  3  3  3

□  □  □  □  4  4  4  4  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5

X  X  X  X  ✓  X  ✓  ✓  ✓  X  X

☐  1  1  1  2  2  2  2  2  2  1  5

☐  ☐  5  5  5  5  3  3  3  3  3  3

☐  ☐  ☐  4  4  4  4  4  4  4  4  4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

X X X X ✔ X X ✔ ✔ X X ✔ ✔

[Images and symbols shown]
Recall: Designing and Analyzing Greedy Algorithms

**Greedy Algorithm**
- Build up the solutions in steps
- At each step, make an *irrevocable* decision using a “reasonable” strategy

**Analysis of Greedy Algorithm**
- Safety: Prove that the reasonable strategy is “safe” (key)
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
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**Input:** $k$: the size of cache

$n$: number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

**Output:** $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit, empty}\} \cup [n]$

- empty stands for an empty page
- “hit” means evicting no pages
Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests
- $p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

**Output:**
- $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit}, \text{empty}\} \cup [n]$
- empty stands for an empty page
- “hit” means evicting no pages
Analysis of Greedy Algorithm

- **Safety**: Prove that the reasonable strategy is “safe” (key)
- **Self-reduce**: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is “safe” (key)
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**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.
Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is “safe” (key)
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$. 
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 

\[
\begin{array}{c}
4 \\
3 \\
2 \\
1 \\
\end{array} \quad \begin{array}{c}
? \\
? \\
? \\
1 \\
\end{array} \quad \begin{array}{c}
? \\
? \\
1 \\
3 \\
\end{array}
\]
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 

The figure shows a cache with pages 1, 2, and 3, and page 4 requested.

In the example, page 2 is evicted at time 1, and page 3 is evicted at time 3.
Proof.

Create $S_0$. $S_0$ evicts $p_{\leftrightarrow}(=3)$ instead of $p_0(=2)$ at time 1.

After time 1, cache status of $S$ and that of $S_0$ differ by only 1 page. $S_0$ contains $p_0(=2)$ and $S$ contains $p_{\leftrightarrow}(=3)$.

From now on, $S_0$ will "copy" $S$.
Proof.

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
Proof.

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
Proof.


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$. 
Proof.


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

6. From now on, $S'$ will “copy” $S$. 

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<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<tbody>
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**S:**

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**S':**

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</table>
**Proof.**


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

6. From now on, $S'$ will “copy” $S$. 

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$S$:

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<tr>
<th></th>
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$S'$:

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<tr>
<td>3</td>
<td></td>
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$\begin{array}{c|c|c}
4 & 5 & \hline
2 & 3 & X
\end{array}$

$\begin{array}{c|c|c}
1 & 1 & \hline
2 & 4 & X
\end{array}$

$\begin{array}{c|c|c}
1 & 1 & \hline
2 & 4 & X
\end{array}$

$\begin{array}{c|c|c}
3 & 3 & \hline
3 & 2 &
\end{array}$
Proof.


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

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6. From now on, $S'$ will “copy” $S$. 

\[
\begin{array}{ccc}
1 & 1 & 5 \\
2 & 4 & 4 \\
3 & 3 & 3
\end{array}
\] 

\[
\begin{array}{ccc}
1 & 1 & 5 \\
2 & 4 & 4 \\
3 & 2 & 2
\end{array}
\]
Proof.

4 Create $S'$. $S'$ evicts $p^* (=3)$ instead of $p' (=2)$ at time 1.

5 After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p' (=2)$ and $S$ contains $p^* (=3)$.

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5 After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

6 From now on, $S'$ will “copy” $S$. 
Proof.

If $S$ evicted the page $p^{\star}$, $S_0$ will evict the page $p_0^{\star}$. Then, the cache status of $S$ and that of $S_0$ will be the same. $S$ and $S_0$ will be exactly the same from now on.

Assume $S$ did not evict $p^{\star}$ ($=3$) before we see $p_0^{\star}$ ($=2$).
Proof.

If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.
Proof.

7 If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8 Assume $S$ did not evict $p^*(=3)$ before we see $p'(=2)$. 
Proof.

7 If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8 Assume $S$ did not evict $p^*(=3)$ before we see $p'(=2)$. 
Proof.

If $S$ evicts $p_\text{\textasciitilde} (=3)$ for $p_0 (=2)$, then $S$ won't be optimum. Assume otherwise. So far, $S_0$ has 1 less page-miss than $S$ does. The status of $S_0$ and that of $S$ only differ by 1 page.

**$S$:**

```
1 1 5 5 5 ... 6
2 4 4 4 8
3 3 3 3 3 ...
```

**$S'$:**

```
1 1 5 5 5 ... 6
2 4 4 4 8
3 2 2 2 2 ...
```
Proof.

If \( S \) evicts \( p^\leftrightarrow(=3) \) for \( p_0(=2) \), then \( S \) won't be optimum. Assume otherwise.

So far, \( S_0 \) has 1 less page-miss than \( S \) does.

The status of \( S_0 \) and that of \( S \) only differ by 1 page.
Proof.

If $S$ evicts $p_0^\ast (=3)$ for $p_0^\ast (=2)$, then $S$ won't be optimum. Assume otherwise. So far, $S_0$ has 1 less page-miss than $S$ does. The status of $S_0$ and that of $S$ only differ by 1 page.
Proof.

If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.
Proof.

If \( S \) evicts \( p^*(=3) \) for \( p'(=2) \), then \( S \) won’t be optimum. Assume otherwise.
Proof.

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If \( S \) evicts \( p^*(=3) \) for \( p'(=2) \), then \( S \) won’t be optimum. Assume otherwise.

---

**Proof.**

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<td>2</td>
<td>( \ldots )</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Proof.

9 If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.

10 So far, $S'$ has 1 less page-miss than $S$ does.
Proof.

9 If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.

10 So far, $S'$ has 1 less page-miss than $S$ does.

11 The status of $S'$ and that of $S$ only differ by 1 page.
Proof.

We can then guarantee that $S_0$ makes at most the same number of page-misses as $S$ does.

Idea: if $S$ has a page-hit and $S_0$ has a page-miss, we use the opportunity to make the status of $S_0$ the same as that of $S$.
Proof. We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.
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We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.

Idea: if $S$ has a page-hit and $S'$ has a page-miss, we use the opportunity to make the status of $S'$ the same as that of $S$.