dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^{\ell}$
    for each $(u, v) \in E$ **do**
        if $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$ **then**
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. $\square$

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

```
f^old[s] ← 0 and f^old[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^old → f^new
    for each (u, v) ∈ E do
        if f^old[u] + w(u, v) < f^new[v] then
            f^new[v] ← f^old[u] + w(u, v)
    copy f^new → f^old
return f^old
```

- f^ℓ only depends on $f^{\ell-1}$: only need 2 vectors

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

$f^{\text{old}}[s] \leftarrow 0$ and $f^{\text{old}}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\text{old}} \to f^{\text{new}}$
    for each $(u, v) \in E$ **do**
        if $f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v]$ **then**
            $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$
    copy $f^{\text{new}} \to f^{\text{old}}$
return f^{old}

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

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  copy $f \rightarrow f$
  for each $(u, v) \in E$ **do**
    if $f[u] + w(u, v) < f[v]$ **then**
      $f[v] \leftarrow f[u] + w(u, v)$
  copy $f \rightarrow f$
return f

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Dynamic Programming with Better Space Usage

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Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

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- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration

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- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges

Bellman-Ford Algorithm

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            f[v] ← f[u] + w(u, v)
return f
```

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

Bellman-Ford Algorithm

- After iteration ℓ:

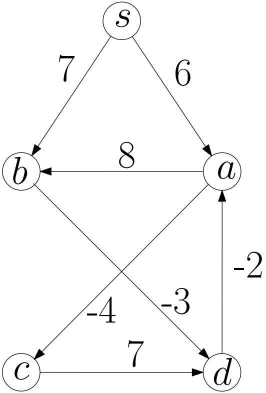
$$\begin{aligned} &\text{length of shortest } s\text{-}v \text{ path} \\ \leq\ & f[v] \\ \leq\ & \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

- Assuming there are no negative cycles:

$$\begin{aligned} &\text{length of shortest } s\text{-}v \text{ path} \\ =\ & \text{length of shortest } s\text{-}v \text{ path using at most } n-1 \text{ edges} \end{aligned}$$

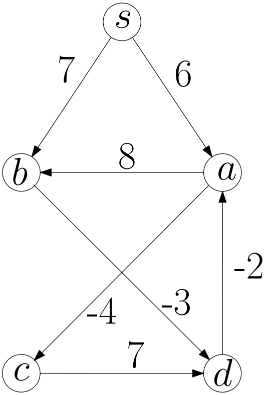
- So, assuming there are no negative cycles, after iteration $n-1$:

$$f[v] = \text{length of shortest } s\text{-}v \text{ path}$$



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



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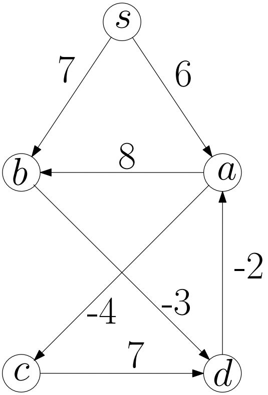
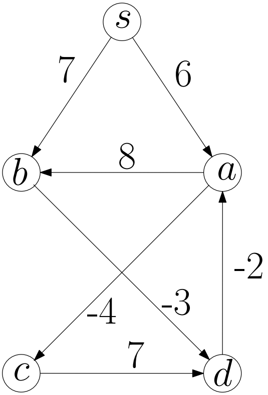
vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



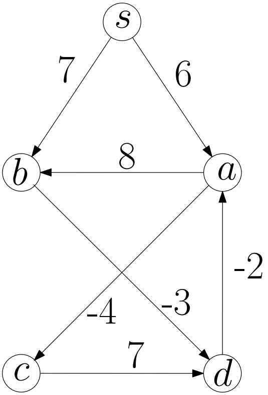
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	∞	∞	∞



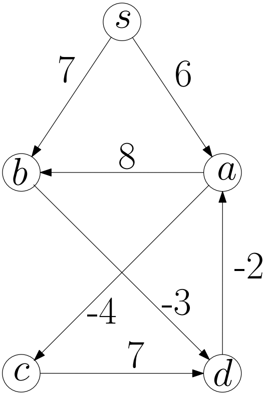
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	∞	∞	∞



- order in which we consider edges:
  (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



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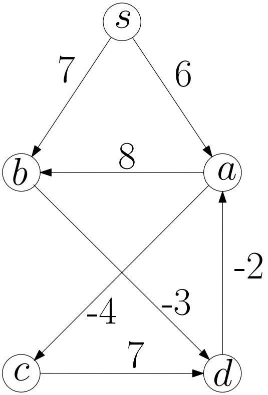
vertices	s	a	b	c	d
f	0	6	7	∞	∞



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vertices	s	a	b	c	d
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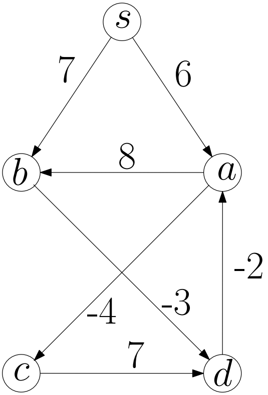
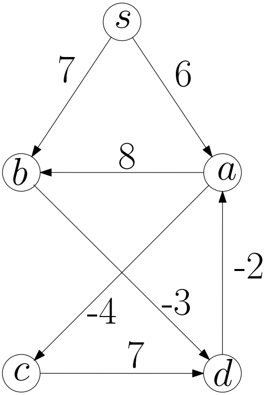


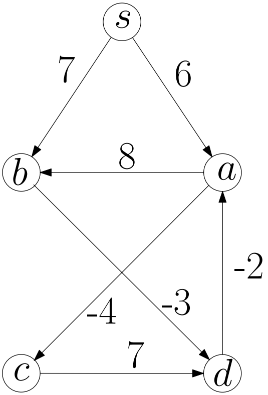
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	∞



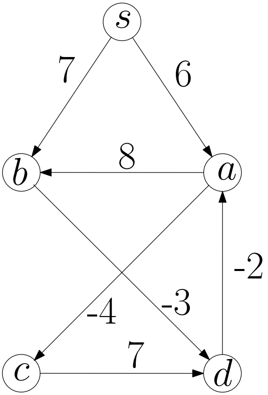
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	7	2	∞



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	4



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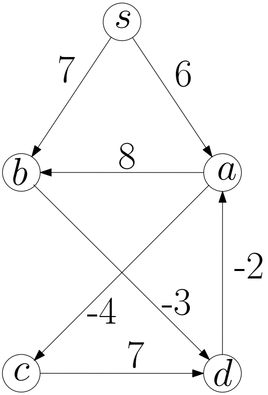

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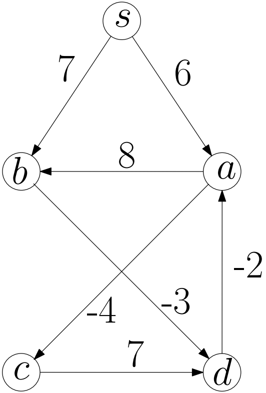

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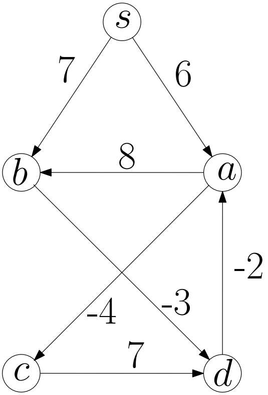

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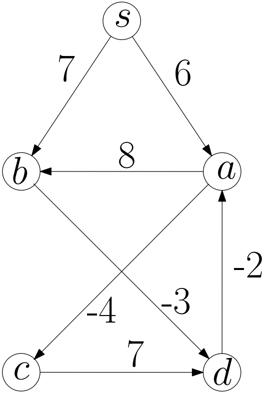
- end of iteration 1: 0, 2, 7, 2, 4



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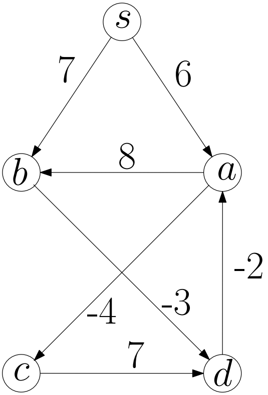

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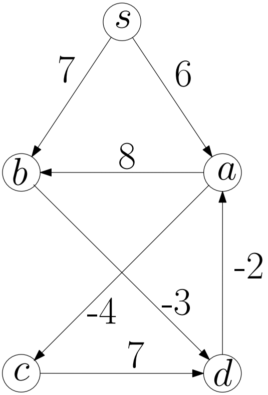

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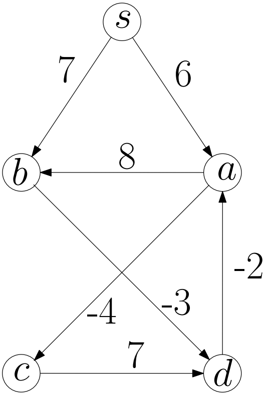

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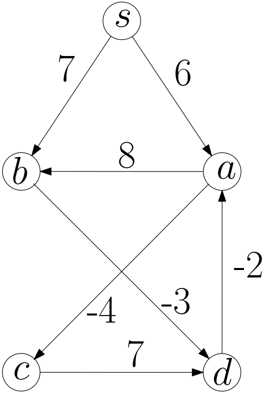

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vertices	s	a	b	c	d
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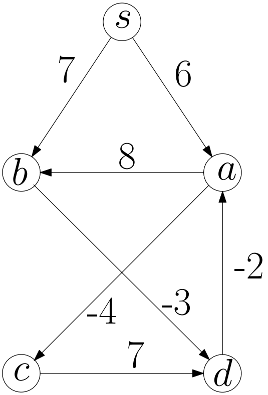

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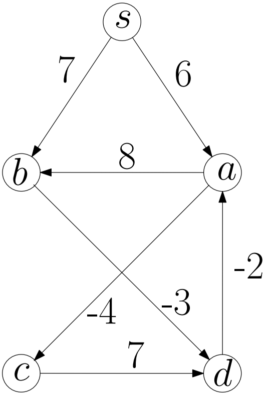

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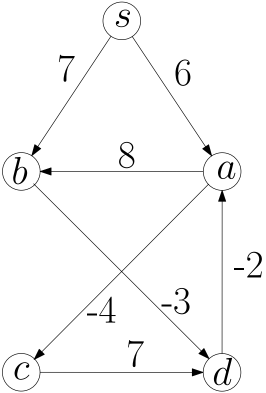

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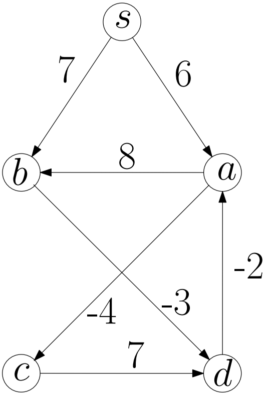

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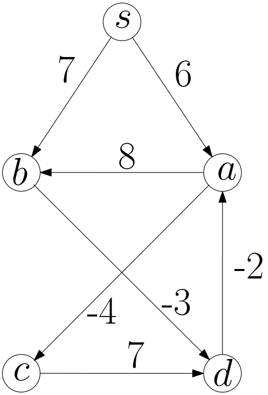
- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4



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vertices	s	a	b	c	d
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- end of iteration 1: 0, 2, 7, 2, 4
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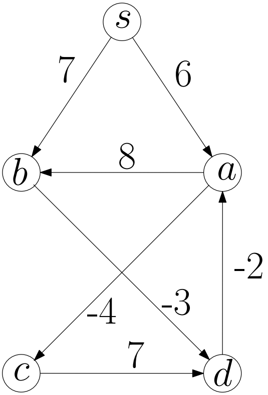
- end of iteration 3: 0, 2, 7, -2, 4



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vertices	s	a	b	c	d
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- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4
- Algorithm terminates in 3 iterations, instead of 4.

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to n **do**
    $updated \leftarrow$ false
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$
            $updated \leftarrow$ true
    if not $updated$, then return f
output "negative cycle exists"

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            f[v] ← f[u] + w(u, v), π[v] ← u
            updated ← true
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```

- $\pi[v]$: the parent of v in the shortest path tree

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    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
            $updated \leftarrow$ true
    if not $updated$, then return f
output "negative cycle exists"

- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

All-Pair Shortest Paths

All Pair Shortest Paths

Input: directed graph $G = (V, E)$,
$w : E \to \mathbb{R}$ (can be negative)

Output: shortest path from u to v for every $u, v \in V$

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1: **for** every starting point $s \in V$ **do**
2: run Bellman-Ford(G, w, s)

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Output: shortest path from u to v for every $u, v \in V$

1: **for** every starting point $s \in V$ **do**
2: run Bellman-Ford(G, w, s)

- Running time $= O(n^2 m)$

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n\log n+m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs

Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$

Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
- For simplicity, extend the w values to non-edges:

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

Design a Dynamic Programming Algorithm

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- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices