When we talk about upper bound on running time:

- **Logarithmic time:** $O(\log n)$
- **Linear time:** $O(n)$
- **Quadratic time** $O(n^2)$
- **Cubic time** $O(n^3)$
- **Polynomial time:** $O(n^k)$ for some constant $k$
  
  - $O(n \log n) \subseteq O(n^{1.1})$. So, an $O(n \log n)$-time algorithm is also a polynomial time algorithm.
- **Exponential time:** $O(c^n)$ for some $c > 1$
- **Sub-linear time:** $o(n)$
- **Sub-quadratic time:** $o(n^2)$
Goal of Algorithm Design

Design algorithms to minimize the order of the running time.
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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.
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- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.

- Makes our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Q: Does ignoring the leading constant cause any issues?

- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?
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A:
- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
- So, for reasonably large $n$, algorithm with lower order running time beats algorithm with higher order running time.
Graph Basics

Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo
Outline

1. Graphs

2. Connectivity and Graph Traversal
   - Types of Graphs

3. Bipartite Graphs
   - Testing Bipartiteness

4. Topological Ordering
Examples of Graphs

Figure: Road Networks

Figure: Social Networks

Figure: Internet

Figure: Transition Graphs
(Undirected) Graph $G = (V, E)$

- $V$: set of vertices (nodes);
- $E$: pairwise relationships among $V$;
  - (undirected) graphs: relationship is symmetric, $E$ contains subsets of size 2
(Undirected) Graph \( G = (V, E) \)

- **\( V \):** set of vertices (nodes);
  \[ V = \{1, 2, 3, 4, 5, 6, 7, 8\} \]

- **\( E \):** pairwise relationships among \( V \);
  - (undirected) graphs: relationship is symmetric, \( E \) contains subsets of size 2
  \[ E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\} \]
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  - $E = \{(1, 2), (1, 3), (3, 2), (4, 2), (2, 5), (5, 3), (3, 7), (3, 8), (4, 5), (5, 6), (6, 5), (8, 7)\}$
Abuse of Notations

- For (undirected) graphs, we often use \((i, j)\) to denote the set \(\{i, j\}\).
- We call \((i, j)\) an unordered pair; in this case \((i, j) = (j, i)\).

\[ E = \{(1, 2), (1, 3), (2, 3), (2, 4), (2, 5), (3, 5), (3, 7), (3, 8), (4, 5), (5, 6), (7, 8)\} \]
- Social Network: Undirected
- Transition Graph: Directed
- Road Network: Directed or Undirected
- Internet: Directed or Undirected
Adjacency matrix
- \( n \times n \) matrix, \( A[u, v] = 1 \) if \((u, v) \in E\) and \( A[u, v] = 0 \) otherwise
- \( A \) is symmetric if graph is undirected
Representation of Graphs

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Linked lists
- For every vertex \( v \), there is a linked list containing all neighbors of \( v \).
Representation of Graphs

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- Linked lists
  - For every vertex $v$, there is a linked list containing all neighbors of $v$.
  - When graph is static, can use array of variant-length arrays.
Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- \( n \): number of vertices
- \( m \): number of edges, assuming \( n - 1 \leq m \leq n(n - 1)/2 \)
- \( d_v \): number of neighbors of \( v \)

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Connectivity Problem

**Input:** graph $G = (V, E)$, (using linked lists)

two vertices $s, t \in V$

**Output:** whether there is a path connecting $s$ to $t$ in $G$
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Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- $L_{j+1}$ contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in $L_j$
Breadth-First Search (BFS)

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Implementing BFS using a Queue

**BFS(s)**

1. \( head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s \)
2. mark \( s \) as “visited” and all other vertices as “unvisited”
3. while \( head \leq tail \) do
4. \( v \leftarrow queue[head], head \leftarrow head + 1 \)
5. for all neighbors \( u \) of \( v \) do
6. if \( u \) is “unvisited” then
7. \( tail \leftarrow tail + 1, queue[tail] = u \)
8. mark \( u \) as “visited”

- Running time: \( O(n + m) \).
Example of BFS via Queue
Example of BFS via Queue
Example of BFS via Queue
Example of BFS via Queue

![BFS via Queue Diagram]

The diagram on the left illustrates a graph with nodes labeled from 1 to 8, connecting with edges to form a network. The node v is highlighted, indicating the starting point for a Breadth-First Search (BFS) traversal. The diagram on the right shows a queue representation with elements 1, 2, and 3, illustrating the order in which nodes are processed during the BFS. The queue is marked with 'head' at the top and 'tail' at the bottom, depicting the insertion and removal of nodes.
Example of BFS via Queue

Graph:
- Nodes: 1, 2, 3, 4, 5, 6, 7, 8
- Edges: 1-2, 1-3, 2-3, 2-4, 2-5, 3-7, 5-8

Queue:
- Head
- Tail

Algorithm:
1. Enqueue node v
2. Dequeue node from queue
3. Process node
4. Enqueue all unvisited neighbors of dequeued node
5. Repeat until queue is empty

Example:
- Start with node v
- Enqueue v
- Dequeue v
- Process v
- Enqueue neighbors of v
- Repeat until all nodes are processed
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![Graph and Queue Diagram]
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