Outline

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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

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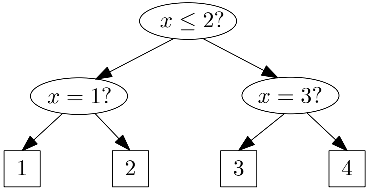

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Q: How many questions do you need to ask in order to get the permutation π?

A: At least $\log_2 n! = \Theta(n \lg n)$

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Selection Problem

Input: a set A of n numbers, and $1 \leq i \leq n$

Output: the i-th smallest number in A

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- Sorting solves the problem in time $O(n \lg n)$.
- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

1: **if** $n \leq 1$ **then return** A
2: $x \leftarrow$ lower median of A
3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
5: $B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ ▷ Conquer
6: $B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ ▷ Conquer
7: $t \leftarrow$ number of times x appear A
8: **return** the array obtained by concatenating B_L, the array containing t copies of x, and B_R

Selection Algorithm with Median Finder

selection(A, n, i)

if $n = 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
$A_R \leftarrow$ elements in A that are greater than x ▷ Divide
if $i \leq A_L$.size **then**
  return selection$(A_L, A_L.\text{size}, i)$ ▷ Conquer
else if $i > n - A_R$.size **then**
  return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ ▷ Conquer
else
  return x

Selection Algorithm with Median Finder

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

if $n = 1$ **thenreturn** A
$x \leftarrow$ random element of A (called pivot)
$A_L \leftarrow$ elements in A that are less than x ▷ Divide
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if $i \leq A_L.\text{size}$ **then**
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- expected running time $= O(n)$

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Input: two polynomials of degree $n-1$

Output: product of two polynomials

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$$(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5)$$

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Example:

$$
\begin{aligned}
&(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\
= &\, 6x^6 - 9x^5 + 18x^4 - 15x^3 \\
&+ 4x^5 - 6x^4 + 12x^3 - 10x^2 \\
&- 10x^4 + 15x^3 - 30x^2 + 25x \\
&+ 8x^3 - 12x^2 + 24x - 20 \\
= &\, 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}
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=\ & 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}$$

- **Input**: $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output**: $(-20, 49, -52, 20, 2, -5, 6)$

Naïve Algorithm

polynomial-multiplication(A, B, n)

let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
        $C[i+j] \leftarrow C[i+j] + A[i] \times B[j]$
return C

Naïve Algorithm

polynomial-multiplication(A, B, n)

```
let C[k] ← 0 for every k = 0, 1, 2, ··· , 2n − 2
for i ← 0 to n − 1 do
    for j ← 0 to n − 1 do
        C[i + j] ← C[i + j] + A[i] × B[j]
return C
```

Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

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- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

$$pq = \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right)$$