Polynomial Multiplication

Input: two polynomials of degree $n-1$

Output: product of two polynomials

Example:

$$
\begin{aligned}
& (3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\
= & \; 6x^6 - 9x^5 + 18x^4 - 15x^3 \\
& + 4x^5 - 6x^4 + 12x^3 - 10x^2 \\
& - 10x^4 + 15x^3 - 30x^2 + 25x \\
& + 8x^3 - 12x^2 + 24x - 20 \\
= & \; 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}
$$

- **Input**: $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output**: $(-20, 49, -52, 20, 2, -5, 6)$

Naïve Algorithm

polynomial-multiplication(A, B, n)

let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
for $i \leftarrow 0$ to $n-1$ **do**
  for $j \leftarrow 0$ to $n-1$ **do**
    $C[i+j] \leftarrow C[i+j] + A[i] \times B[j]$
return C

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Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

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- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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$$pq = \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right)$$

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$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right)x^{n/2} + p_L q_L \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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$$

$$
\begin{aligned}
\mathsf{multiply}(p, q) = {} & \mathsf{multiply}(p_H, q_H) \times x^n \\
& + \big(\mathsf{multiply}(p_H, q_L) + \mathsf{multiply}(p_L, q_H)\big) \times x^{n/2} \\
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- Recurrence: $T(n) = 4T(n/2) + O(n)$

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- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

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&= p_H q_H x^n + \big(p_H q_L + p_L q_H\big) x^{n/2} + p_L q_L
\end{aligned}
$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

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$$r_H = \mathsf{multiply}(p_H, q_H)$$
$$r_L = \mathsf{multiply}(p_L, q_L)$$

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \mathsf{multiply}(p_H, q_H)$$
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$$\begin{aligned}\mathsf{multiply}(p, q) = {} & r_H \times x^n \\ & + \big(\mathsf{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L\big) \times x^{n/2} \\ & + r_L\end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \mathrel{..} n/2-1], A_H \leftarrow A[n/2 \mathrel{..} n-1]$
$B_L \leftarrow B[0 \mathrel{..} n/2-1], B_H \leftarrow B[n/2 \mathrel{..} n-1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n-1)$ 0's
for $i \leftarrow 0$ to $n-2$ **do**
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i+n] \leftarrow C[i+n] + C_H[i]$
    $C[i+n/2] \leftarrow C[i+n/2] + C_M[i] - C_L[i] - C_H[i]$
return C

Outline

1 Divide-and-Conquer

2 Counting Inversions

3 Quicksort and Selection
- Quicksort
- Lower Bound for Comparison-Based Sorting Algorithms
- Selection Problem

4 Polynomial Multiplication

5 Solving Recurrences

6 Computing n-th Fibonacci Number

7 Other Classic Algorithms using Divide-and-Conquer

Methods for Solving Recurrences

- The recursion-tree method
- The master theorem

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$

Recursion-Tree Method

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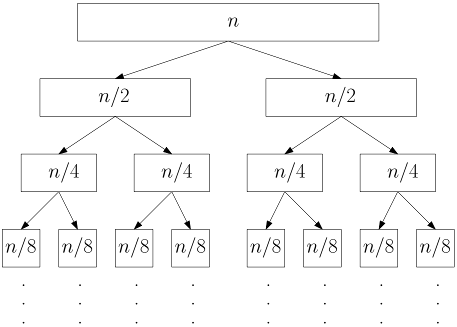

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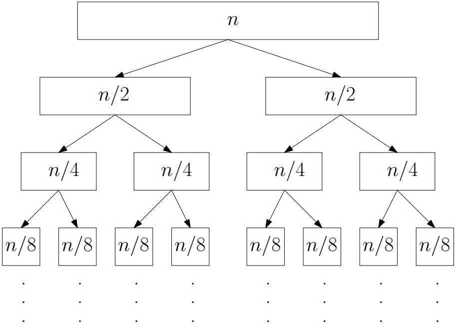

- Each level takes running time $O(n)$

Recursion-Tree Method

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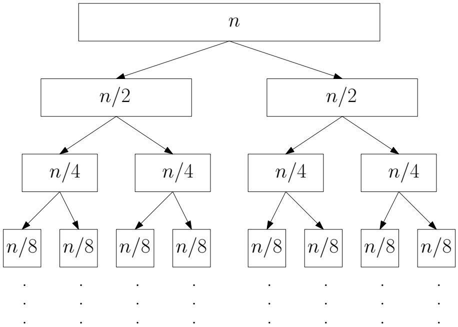

- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

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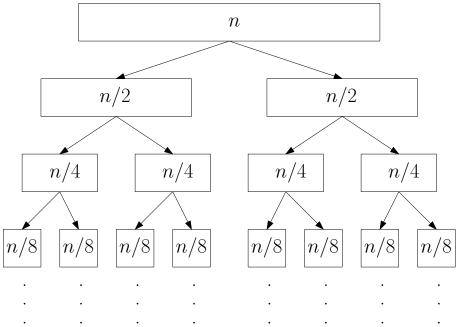

- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

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Recursion-Tree Method

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n

Recursion-Tree Method

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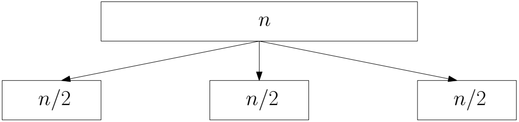

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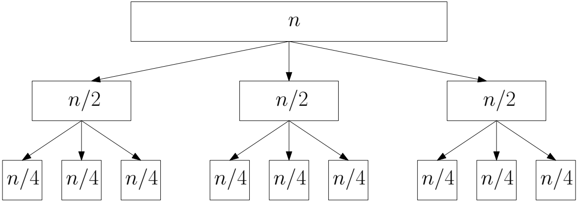

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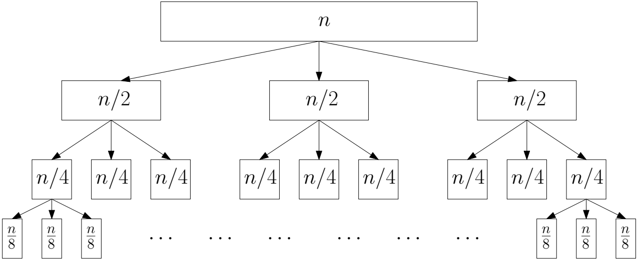

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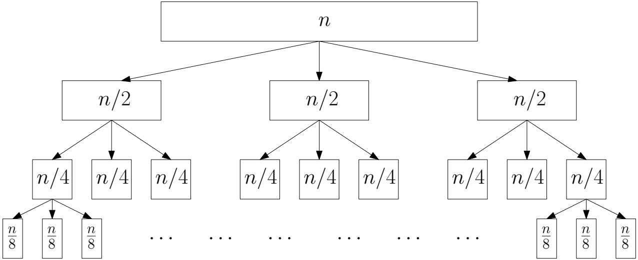

- Total running time at level i?

Recursion-Tree Method

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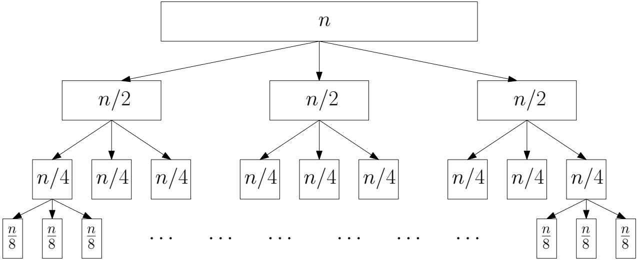

- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$

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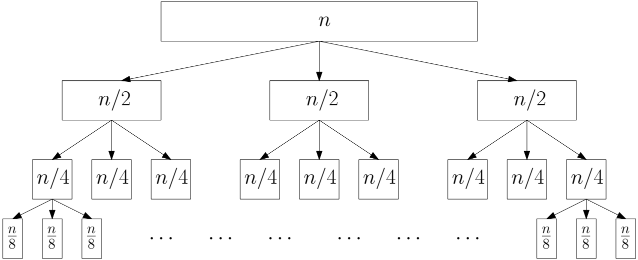

- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
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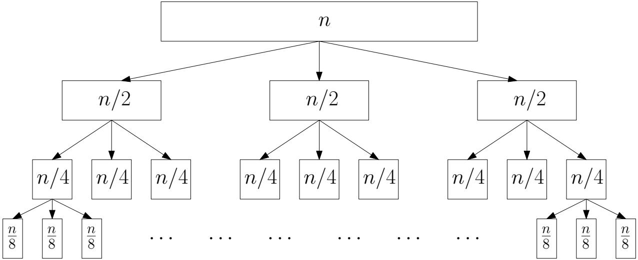

- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
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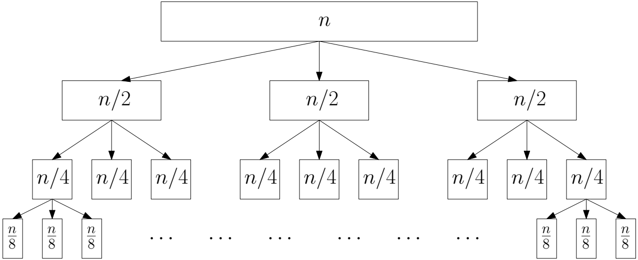

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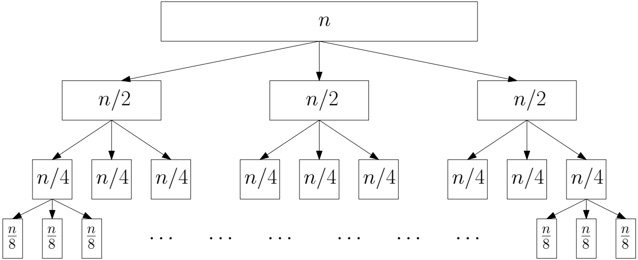

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- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n\left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

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n^2

Recursion-Tree Method

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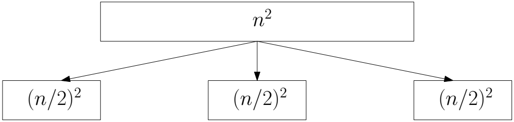

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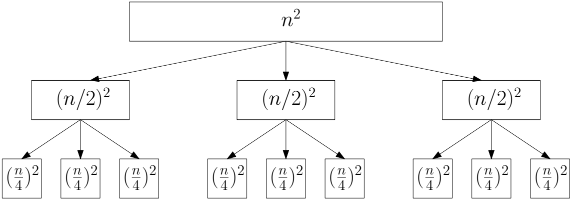

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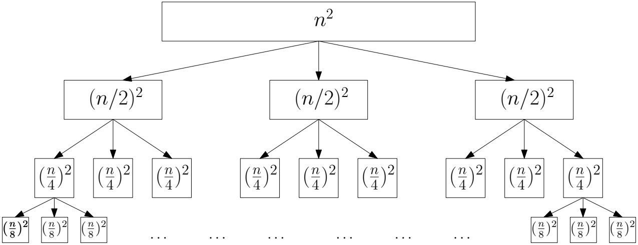

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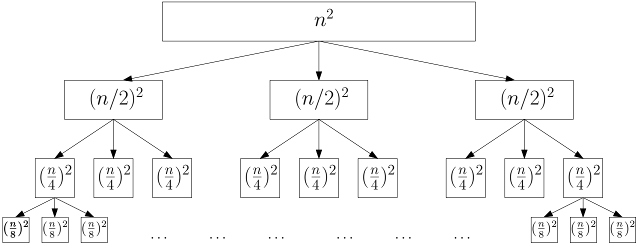

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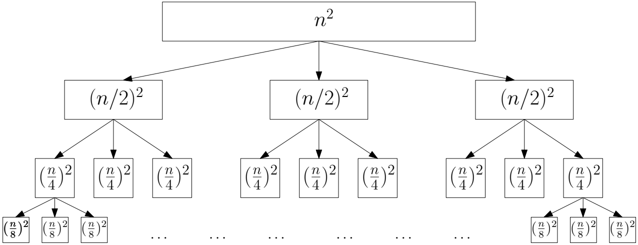

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$

Recursion-Tree Method

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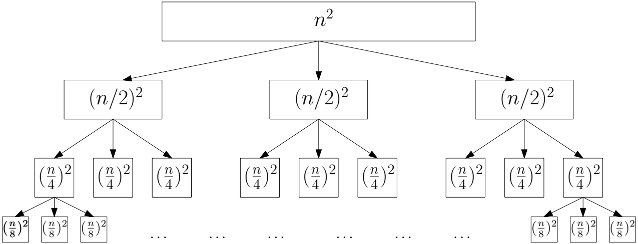

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level?

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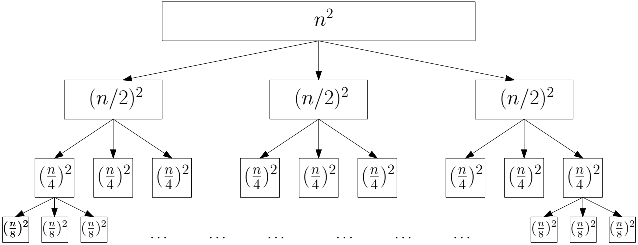

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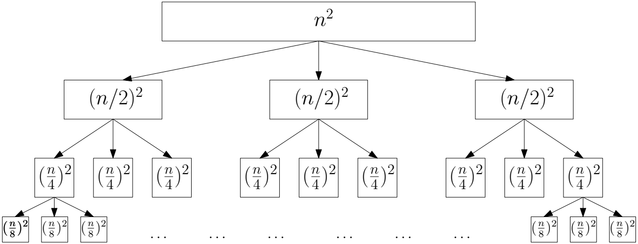

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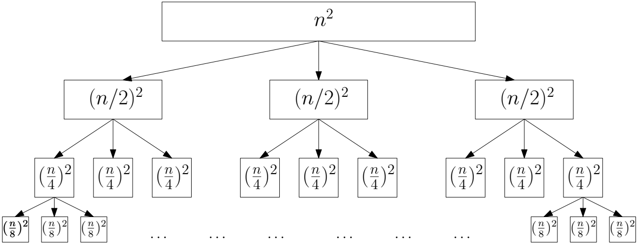

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$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 =$$

Recursion-Tree Method

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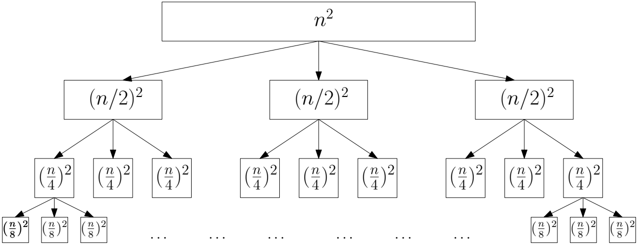

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- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$				$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
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$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
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Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$	3	2	2	$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
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- Ex: $T(n) = T(n/2) + O(1)$. Case 2. $T(n) = O(\lg n)$
- Ex: $T(n) = 2T(n/2) + O(n^2)$. Case 3.

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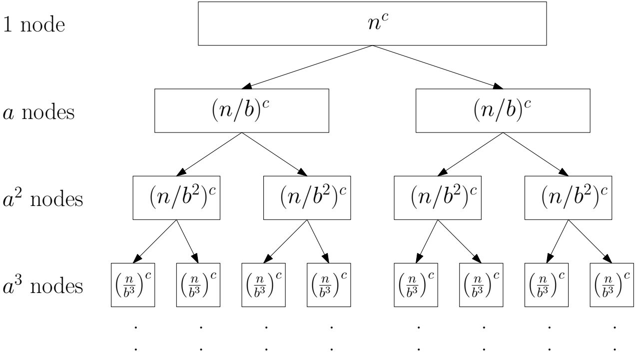
Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



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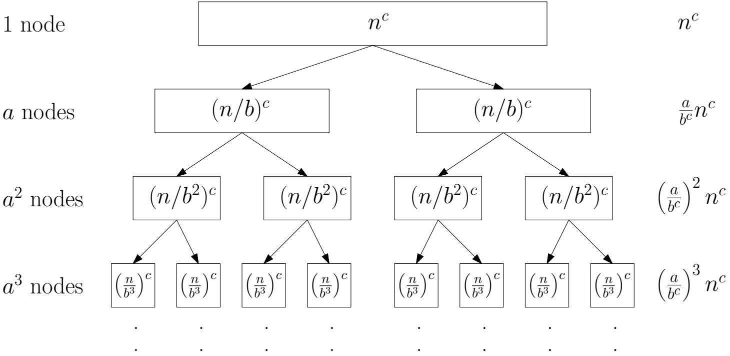

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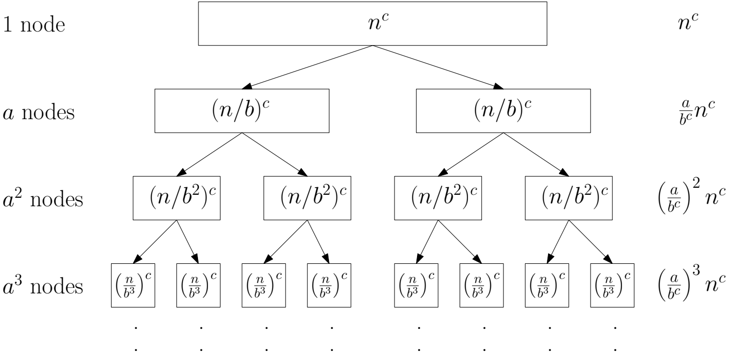

- $c < \lg_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\lg_b n} n^c = n^{\lg_b a}$

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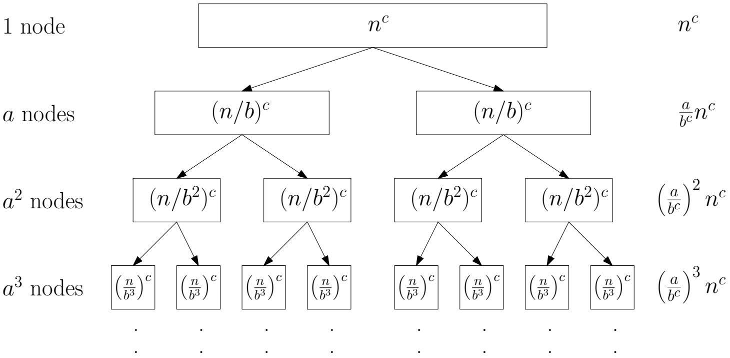

- $c < \lg_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\lg_b n} n^c = n^{\lg_b a}$
- $c = \lg_b a$: all levels have same time: $n^c \lg_b n = O(n^c \lg n)$

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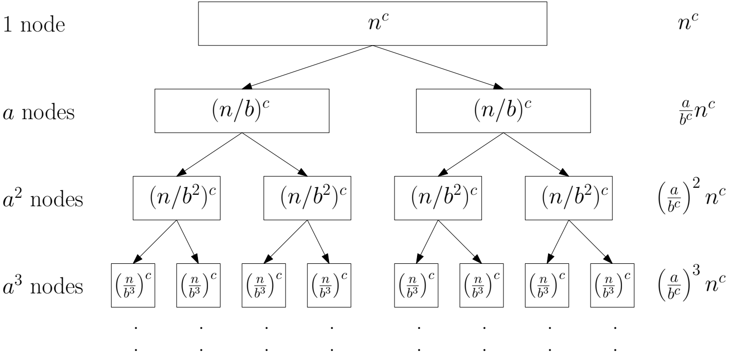

- $c < \lg_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\lg_b n} n^c = n^{\lg_b a}$
- $c = \lg_b a$: all levels have same time: $n^c \lg_b n = O(n^c \lg n)$
- $c > \lg_b a$: top-level dominates: $O(n^c)$

Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
   - Quicksort
   - Lower Bound for Comparison-Based Sorting Algorithms
   - Selection Problem
4. Polynomial Multiplication
5. Solving Recurrences
6. Computing n-th Fibonacci Number
7. Other Classic Algorithms using Divide-and-Conquer

Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

n-th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

```
if n = 0 return 0
if n = 1 return 1
return Fib(n − 1) + Fib(n − 2)
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Q: Is the running time of the algorithm polynomial or exponential in n?

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A: Exponential