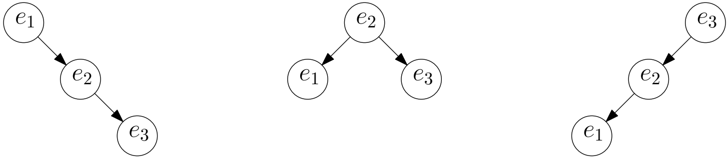
Optimum Binary Search Tree

- n elements $e_1 < e_2 < e_3 < \cdots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \cdots, e_n\}$ with the minimum accessing cost:

$$\sum_{i=1}^{n} f_i \times (\text{depth of } e_i \text{ in the tree})$$

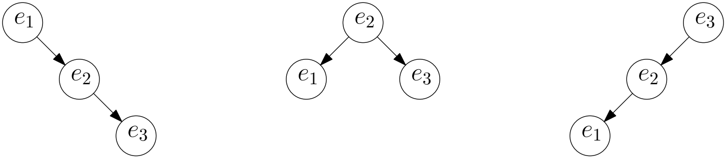
Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



Optimum Binary Search Tree

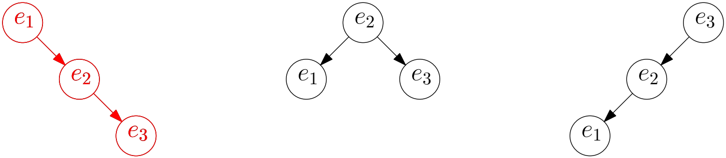
- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

Optimum Binary Search Tree

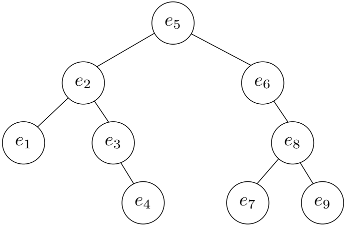
- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

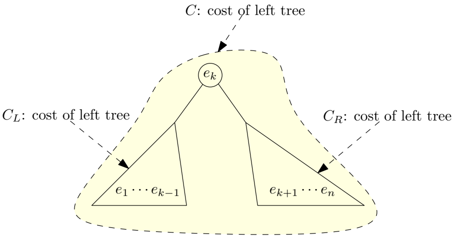
- suppose we decided to let e_k be the root
- $e_1, e_2, \cdots, e_{k-1}$ are on left sub-tree
- $e_{k+1}, e_{k+2}, \cdots, e_n$ are on right sub-tree

- d_j: depth of e_j in our tree
- C, C_L, C_R: cost of tree, left sub-tree and right sub-tree



- $d_1 = 3, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 1,$
- $d_6 = 2, d_7 = 4, d_8 = 3, d_9 = 4,$
- $C = 3f_1 + 2f_2 + 3f_3 + 4f_4 + f_5 + 2f_6 + 4f_7 + 3f_8 + 4f_9$
- $C_L = 2f_1 + f_2 + 2f_3 + 3f_4$
- $C_R = f_6 + 3f_7 + 2f_8 + 3f_9$

- $C = C_L + C_R + \sum_{j=1}^{9} f_j$



$$
\begin{aligned}
C = \sum_{\ell=1}^{n} f_\ell d_\ell &= \sum_{\ell=1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= \sum_{\ell=1}^{k-1} f_\ell (d_\ell - 1) + \sum_{\ell=k+1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= C_L + C_R + \sum_{\ell=1}^{n} f_\ell
\end{aligned}
$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$opt[1, n] = \qquad (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$opt[1, n] = \min_{k: 1 \le k \le n} (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$opt[1, n] = \min_{k:1 \leq k \leq n} \left(opt[1, k-1] + opt[k+1, n]\right) + \sum_{\ell=1}^{n} f_\ell$$

- In general, $opt[i, j] =$

$$\begin{cases} 0 & \text{if } i = j+1 \\ \min_{k:i \leq k \leq j} \big(opt[i, k-1] + opt[k+1, j]\big) + \sum_{\ell=i}^{j} f_\ell & \text{if } i \leq j \end{cases}$$

Optimum Binary Search Tree

$fsum[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do** $fsum[i] \leftarrow fsum[i-1] + f_i$ $\triangleright\ fsum[i] = \sum_{j=1}^{i} f_j$
for $i \leftarrow 0$ to n **do** $opt[i+1, i] \leftarrow 0$
for $\ell \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to $n - \ell + 1$ **do**
        $j \leftarrow i + \ell - 1$, $opt[i, j] \leftarrow \infty$
        for $k \leftarrow i$ to j **do**
            if $opt[i, k-1] + opt[k+1, j] < opt[i, j]$ **then**
                $opt[i, j] \leftarrow opt[i, k-1] + opt[k+1, j]$
                $\pi[i, j] \leftarrow k$
        $opt[i, j] \leftarrow opt[i, j] + fsum[j] - fsum[i-1]$

Printing the Tree

Print-Tree(i, j)

if $i > j$ **then**
    return
else
    print('(')
    Print-Tree$(i, \pi[i, j] - 1)$
    print$(\pi[i, j])$
    Print-Tree$(\pi[i, j] + 1, j)$
    print(')')

Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree: $opt[i, j]$ = value of instances defined by matrices i to j

Outline

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:
  - Undirected graph, directed graph

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:
  - Undirected graph, directed graph
  - Two representations: adjacency matrix, linked lists

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:
  - Undirected graph, directed graph
  - Two representations: adjacency matrix, linked lists
  - Path, cycle, tree, directed acyclic graph, bipartite graph

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:
  - Undirected graph, directed graph
  - Two representations: adjacency matrix, linked lists
  - Path, cycle, tree, directed acyclic graph, bipartite graph
  - Connectivity problem: BFS and DFS algorithm

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:
  - Undirected graph, directed graph
  - Two representations: adjacency matrix, linked lists
  - Path, cycle, tree, directed acyclic graph, bipartite graph
  - Connectivity problem: BFS and DFS algorithm
  - Testing Bipartiteness problem: test-bipartiteness-BFS or test-bipartiteness-DFS algorithm

Important notations/algorithms

- Introduction:
  - Asymptotic analysis: O, Ω, Θ, compare the orders
  - Polynomial time (efficient algorithm), exponential time
- Graph Basics:
  - Undirected graph, directed graph
  - Two representations: adjacency matrix, linked lists
  - Path, cycle, tree, directed acyclic graph, bipartite graph
  - Connectivity problem: BFS and DFS algorithm
  - Testing Bipartiteness problem: test-bipartiteness-BFS or test-bipartiteness-DFS algorithm
  - Topological Ordering problem: topological-sort algorithm

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm
  - Interval Scheduling problem: schedule algorithm

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm
  - Interval Scheduling problem: schedule algorithm
  - Interval Partitioning problem: partition algorithm

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm
  - Interval Scheduling problem: schedule algorithm
  - Interval Partitioning problem: partition algorithm
  - Offline Caching problem: FIF algorithm

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm
  - Interval Scheduling problem: schedule algorithm
  - Interval Partitioning problem: partition algorithm
  - Offline Caching problem: FIF algorithm
  - Priority Queue: heap

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm
  - Interval Scheduling problem: schedule algorithm
  - Interval Partitioning problem: partition algorithm
  - Offline Caching problem: FIF algorithm
  - Priority Queue: heap
  - Huffman Code problem: prefix code notation, Huffman algorithm

Important notations/algorithms

- Greedy algorithms: safety strategy+self reduce
  - Box Packing problem: greedy algorithm
  - Interval Scheduling problem: schedule algorithm
  - Interval Partitioning problem: partition algorithm
  - Offline Caching problem: FIF algorithm
  - Priority Queue: heap
  - Huffman Code problem: prefix code notation, Huffman algorithm
  - Exercise problems: Fractional knapsack problem, scheduling problem (min weighted sum of completion times)

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
  - Counting inversions problem: sort-and-count algorithm

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
  - Counting inversions problem: sort-and-count algorithm
  - Selection problem: selection algorithm based on quicksort

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
  - Counting inversions problem: sort-and-count algorithm
  - Selection problem: selection algorithm based on quicksort
  - Polynomial Multiplication problem: multiply algorithm

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
  - Counting inversions problem: sort-and-count algorithm
  - Selection problem: selection algorithm based on quicksort
  - Polynomial Multiplication problem: multiply algorithm
  - Recurrences: recursive-tree method and Master Theorem

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
  - Counting inversions problem: sort-and-count algorithm
  - Selection problem: selection algorithm based on quicksort
  - Polynomial Multiplication problem: multiply algorithm
  - Recurrences: recursive-tree method and Master Theorem
  - Fibonacci number problem: power algorithm

Important notations/algorithms

- Divide-and-Conquer algorithms: Divide+Conquer+Combine
  - Sorting problem: merge-sort algorithm, quick-sort algorithm (and In-Place sorting algorithm)
  - Counting inversions problem: sort-and-count algorithm
  - Selection problem: selection algorithm based on quicksort
  - Polynomial Multiplication problem: multiply algorithm
  - Recurrences: recursive-tree method and Master Theorem
  - Fibonacci number problem: power algorithm
  - Exercise problems: Closest Pair, Convex Hull, Two Matrix Multiplication

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule
  - Longest common subsequence problem (LCS): DP algorithm + Recovering optimal schedule

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule
  - Longest common subsequence problem (LCS): DP algorithm + Recovering optimal schedule
  - Edit distance with insertions and deletions problem: apply algorithm for LCS problem

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule
  - Longest common subsequence problem (LCS): DP algorithm + Recovering optimal schedule
  - Edit distance with insertions and deletions problem: apply algorithm for LCS problem
  - Edit distance with insertions, deletions and replacing problem

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule
  - Longest common subsequence problem (LCS): DP algorithm + Recovering optimal schedule
  - Edit distance with insertions and deletions problem: apply algorithm for LCS problem
  - Edit distance with insertions, deletions and replacing problem
  - Shortest Path in Directed Acyclic Graph (DAG): Shortest Paths in DAG algorithm + print-path algorithm

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule
  - Longest common subsequence problem (LCS): DP algorithm + Recovering optimal schedule
  - Edit distance with insertions and deletions problem: apply algorithm for LCS problem
  - Edit distance with insertions, deletions and replacing problem
  - Shortest Path in Directed Acyclic Graph (DAG): Shortest Paths in DAG algorithm + print-path algorithm
  - Matrix Chain Multiplication problem: matrix-chain-multiplication algorithm + print-optimal-order alg

Important notations/algorithms

- Dynamic Programming algorithms: subproblem+recurrence relation+calculate from base case
  - Weighted interval scheduling problem: DP algorithm + Recovering optimal schedule
  - Subset Sum problem: DP algorithm + Recovering optimal schedule
  - Knapsack problem: DP algorithm + Recovering optimal schedule
  - Longest common subsequence problem (LCS): DP algorithm + Recovering optimal schedule
  - Edit distance with insertions and deletions problem: apply algorithm for LCS problem
  - Edit distance with insertions, deletions and replacing problem
  - Shortest Path in Directed Acyclic Graph (DAG): Shortest Paths in DAG algorithm + print-path algorithm
  - Matrix Chain Multiplication problem: matrix-chain-multiplication algorithm + print-optimal-order alg
  - Optimum Binary Search Tree Problem: Optimum Binary Search Tree alg + Print Tree alg

Quiz 6 about Dynamic Programming algorithms

- Fours problems about Dynamic programming algorithms