Reductions of NP-Complete Problems

- Circuit-Sat
  - 3-Sat
    - Clique
    - Ind-Set
      - Vertex-Cover
        - Set-Cover
    - HC
      - TSP
    - 3D-Matching
      - Subset-Sum
        - Knapsack
    - 3-Coloring
3-CNF (conjunctive normal form) is a special case of formula:
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- Boolean variables: $x_1, x_2, \cdots, x_n$
3-Sat

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- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: $x_i$ or $\neg x_i$
3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: $x_i$ or $\neg x_i$
- Clause: disjunction ("or") of at most 3 literals: $x_3 \lor \neg x_4$, $x_1 \lor x_8 \lor \neg x_9$, $\neg x_2 \lor \neg x_5 \lor x_7$
3-CNF (conjunctive normal form) is a special case of formula:

- **Boolean variables**: \( x_1, x_2, \ldots, x_n \)
- **Literals**: \( x_i \) or \( \neg x_i \)
- **Clause**: disjunction ("or") of at most 3 literals: \( x_3 \lor \neg x_4, \)
  \( x_1 \lor x_8 \lor \neg x_9, \) \( \neg x_2 \lor \neg x_5 \lor x_7 \)
- **3-CNF formula**: conjunction ("and") of clauses:
  \( (x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor \neg x_4) \)
3-Sat

Input: a 3-CNF formula
Output: whether the 3-CNF is satisfiable
3-Sat

3-Sat

**Input:** a 3-CNF formula

**Output:** whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
3-Sat

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3-Sat

**Input:** a 3-CNF formula

**Output:** whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal
- Assignment $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ satisfies
  
  $$(x_1 \lor \lnot x_2 \lor \lnot x_3) \land (x_2 \lor x_3 \lor x_4) \land (\lnot x_1 \lor \lnot x_3 \lor \lnot x_4)$$
Associate every wire with a new variable

The circuit is equivalent to the following formula:

\((x_4 = \neg x_3)^\wedge (x_5 = x_1 \wedge x_2)^\wedge (x_6 = \neg x_4)^\wedge (x_7 = x_1 \wedge x_2 \wedge x_4)^\wedge (x_8 = x_5 \wedge x_6)^\wedge (x_9 = x_6 \wedge x_7)^\wedge (x_{10} = x_8 \wedge x_9 \wedge x_7)^\wedge x_{10}\)
Circuit-Sat $\leq_P$ 3-Sat

- Associate every wire with a new variable
Associate every wire with a new variable

The circuit is equivalent to the following formula:

\[
\begin{align*}
(x_4 = \neg x_3) & \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4) \\
& \land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6) \\
& \land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}
\end{align*}
\]
$\text{Circuit-Sat } \leq_P 3\text{-Sat}$

\[
\begin{align*}
(x_4 = \neg x_3) & \land (x_5 = x_1 \lor x_2) & \land (x_6 = \neg x_4) \\
\land (x_7 = x_1 \land x_2 \land x_4) & \land (x_8 = x_5 \lor x_6) \\
\land (x_9 = x_6 \lor x_7) & \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}
\end{align*}
\]

Convert each clause to a 3-CNF
Circuit-Sat \( \leq_P \) 3-Sat

\[
(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4) \\
\land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6) \\
\land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}
\]

Convert each clause to a 3-CNF

\[
x_5 = x_1 \lor x_2
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Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4)$$
$$\land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6)$$
$$\land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}$$

Convert each clause to a 3-CNF

$x_5 = x_1 \lor x_2$ $\iff$

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Circuit-Sat \(\leq_P\) 3-Sat

\[
(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4) \\
\land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6) \\
\land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}
\]

Convert each clause to a 3-CNF

\[
x_5 = x_1 \lor x_2 \iff \\
(x_1 \lor x_2 \lor \neg x_5) \land \\
x_5 \leftrightarrow x_1 \lor x_2
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Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4) \land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6) \land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}$$

Convert each clause to a 3-CNF

\[
x_5 = x_1 \lor x_2 \Leftrightarrow
\]

\[
(x_1 \lor x_2 \lor \neg x_5) \land
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Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4)$$
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Convert each clause to a 3-CNF

$$x_5 = x_1 \lor x_2 \iff$$

$$(x_1 \lor x_2 \lor \neg x_5) \land$$

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Circuit-Sat $\leq_P$ 3-Sat

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Convert each clause to a 3-CNF

\[x_5 = x_1 \lor x_2 \quad \iff \quad (x_1 \lor x_2 \lor \neg x_5) \land (x_1 \lor \neg x_2 \lor x_5)\]

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Circuit-Sat $\leq_P$ 3-Sat

\[(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4)\]
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Convert each clause to a 3-CNF

\[x_5 = x_1 \lor x_2 \iff\]

\[(x_1 \lor x_2 \lor \neg x_5) \land\]

\[(x_1 \lor \neg x_2 \lor x_5) \land\]

\[(\neg x_1 \lor x_2 \lor x_5) \land\]

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Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4)$$
$$\land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6)$$
$$\land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \lor x_2 \iff$$
$$\quad (x_1 \lor x_2 \lor \neg x_5) \land$$
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Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4) \land (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6) \land (x_9 = x_6 \lor x_7) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}$$

Convert each clause to a 3-CNF

$x_5 = x_1 \lor x_2$ $\iff$

$$\begin{align*}
(x_1 \lor x_2 \lor \neg x_5) & \land \\
(x_1 \lor \neg x_2 \lor x_5) & \land \\
(\neg x_1 \lor x_2 \lor x_5) & \land \\
(\neg x_1 \lor \neg x_2 \lor x_5) & \land
\end{align*}$$

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Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit
Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit
- Thus, Circuit-Sat $\leq_P$ 3-Sat
Reductions of NP-Complete Problems

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Recall: Independent Set Problem

**Def.** An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.

**Independent Set (Ind-Set) Problem**

**Input:** $G = (V, E), k$

**Output:** whether there is an independent set of size $k$ in $G$
3-Sat \( \leq_P \) Ind-Set

\[
(x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor x_4)
\]
3-Sat $\leq_P$ Ind-Set

- $(x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor x_4)$

- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group
3-Sat $\leq_P$ Ind-Set

- $(x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor x_4)$

- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
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3-Sat \( \leq_P \text{Ind-Set} \)

- \((x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor x_4)\)

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3-Sat instance is yes-instance $\iff$ Ind-Set instance is yes-instance:
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3-Sat instance is yes-instance \iff Ind-Set instance is yes-instance:
- satisfying assignment \Rightarrow independent set of size \(k\)
- independent set of size \(k\) \Rightarrow satisfying assignment
Satisfying Assignment $\Rightarrow$ IS of Size $k$

\[ (x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor x_4) \land (\neg x_1 \lor \neg x_3 \lor x_4) \]
Satisfying Assignment $\Rightarrow$ IS of Size $k$

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IS of Size $k \Rightarrow$ Satisfying Assignment

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- For every group, exactly one literal is selected in IS
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- Otherwise, set $x_i$ arbitrarily
Reductions of NP-Complete Problems

Clique → Ind-Set
  ↓
Vertex-Cover → Set-Cover

Ind-Set

HC

3D-Matching

3-Sat

3-Coloring

Circuit-Sat

TSP

Subset-Sum

Knapsack
**Def.** A **clique** in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$.
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![Graph Illustration](image.png)
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![Clique Diagram]

**Clique Problem**

**Input:** $G = (V, E)$ and integer $k > 0$,  

**Output:** whether there exists a clique of size $k$ in $G$
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Clique Problem

**Input:** $G = (V, E)$ and integer $k > 0,$

**Output:** whether there exists a clique of size $k$ in $G$

- What is the relationship between Clique and Ind-Set?
Clique $= \mathcal{P}$ Ind-Set

**Def.** Given a graph $G = (V, E)$, define $\overline{G} = (V, \overline{E})$ be the graph such that $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.

**Obs.** $S$ is an independent set in $G$ if and only if $S$ is a clique in $\overline{G}$. 
Reductions of NP-Complete Problems

Circuit-Sat
  3-Sat
    Ind-Set
      Clique
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**Vertex-Cover**

**Def.** Given a graph $G = (V, E)$, a vertex cover of $G$ is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$. 

![Graph example](image-url)
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**Vertex-Cover Problem**

**Input:** \( G = (V, E) \) and integer \( k \)

**Output:** whether there is a vertex cover of \( G \) of size at most \( k \)
Vertex-Cover $\equiv_p$ Ind-Set

What is the relationship between Vertex-Cover and Ind-Set?

A: $S$ is a vertex-cover of $G = (V,E)$ if and only if $V \cap S$ is an independent set of $G$. 
Q: What is the relationship between Vertex-Cover and Ind-Set?
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A: \( S \) is a vertex-cover of \( G = (V, E) \) if and only if \( V \setminus S \) is an independent set of \( G \).
Reductions of NP-Complete Problems

Circuit-Sat

3-Sat

Clique

Ind-Set

Vertex-Cover

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3-Coloring
Recall the definition of polynomial time reductions:

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$. 
A Strategy of Polynomial Reduction

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- In general, algorithm for $Y$ can call the algorithm for $X$ many times.
- However, for most reductions, we call algorithm for $X$ only once.
- That is, for a given instance $s_Y$ for $Y$, we only construct one instance $s_X$ for $X$. 
A Strategy of Polynomial Reduction

Given an instance $s_Y$ of problem $Y$, show how to construct in polynomial time an instance $s_X$ of problem such that:

- $s_Y$ is a yes-instance of $Y \implies s_X$ is a yes-instance of $X$
- $s_X$ is a yes-instance of $X \implies s_Y$ is a yes-instance of $Y$
1. Some Hard Problems
2. P, NP and Co-NP
3. Polynomial Time Reductions and NP-Completeness
4. NP-Complete Problems
5. Dealing with NP-Hard Problems
6. Summary
Q: How far away are we from proving or disproving $P = NP$?
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- Try to prove an “unconditional” lower bound on running time of algorithm solving a NP-complete problem.

For 3-Sat problem:

- Assume the number of clauses is $\cdot (n)$,

- $n$ = number variables

- Best algorithm runs in time $O(c^n)$ for some constant $c > 1$

- Best lower bound is $\Omega(n)$

Essentially we have no techniques for proving lower bound for running time
Q: How far away are we from proving or disproving P = NP?

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Dealing with NP-Hard Problems

- Faster exponential time algorithms
- Solving the problem for special cases
- Fixed parameter tractability
- Approximation algorithms
3-SAT:

Brute-force: $O(2^n \cdot \text{poly}(n))$

Practical SAT Solver: solves real-world SAT instances with more than 10,000 variables

Travelling Salesman Problem:

Brute-force: $O(n! \cdot \text{poly}(n))$

Better algorithm: $O(2^n \cdot \text{poly}(n))$

In practice: TSP Solver can solve Euclidean TSP instances with more than 100,000 vertices
Faster Exponential Time Algorithms

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Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on
Solving the problem for special cases

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Approximation Algorithms

For optimization problems, approximation algorithms will find sub-optimal solutions in \textit{polynomial time}.

There is a \(2\)-approximation for the vertex cover problem: we can efficiently find a vertex cover whose size is at most 2 times that of the optimal vertex cover.
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Approximation Algorithms

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- **Approximation ratio** is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution.
- We want to make the approximation ratio as small as possible, while maintaining the property that the algorithm runs in polynomial time.

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There is an 2-approximation for the vertex cover problem: *we can efficiently find a vertex cover whose size is at most 2 times that of the optimal vertex cover*
2-Approximation Algorithm for Vertex Cover

VertexCover($G$)

1: $C \leftarrow \emptyset$
2: while $\neq \emptyset$ do
3: select an edge $(u, v) \in E$, $C \leftarrow C \cup \{u, v\}$
4: Remove from $E$ every edge incident on either $u$ or $v$
5: return $C$

- Let the set $C$ and $C^*$ be the sets output by above algorithm and an optimal alg, respectively. Let $S$ be the set of edges selected.
- Since no two edge in $S$ are covered by the same vertex (Once an edge is picked in line 3, all other edges that are incident on its endpoints are removed from $E$ in line 4), we have $|C^*| \geq |S|$;
- As we have added both vertices of edge $(u, v)$, we get $|C| = 2|S|$ but $C^*$ have to add one of the two, thus, $|C|/|C^*| \leq 2$. 