Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
   - Quicksort
   - Lower Bound for Comparison-Based Sorting Algorithms
   - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. Solving Recurrences
7. Computing n-th Fibonacci Number

Quicksort vs Merge-Sort

	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

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Quicksort

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ array of elements in A that are less than x \\\\ Divide
$A_R \leftarrow$ array of elements in A that are greater than x \\\\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$ \\\\ Conquer
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$t \leftarrow$ number of times x appear A
return concatenation of B_L, t copies of x, and B_R

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A:

1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
2. Choose a pivot randomly and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

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- In practice: use pseudo-random-generator, a deterministic algorithm returning numbers that "look like" random
- In theory: assume they can.

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Lemma The expected running time of the algorithm is $O(n \lg n)$.

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- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

$p \leftarrow$ random integer between ℓ and r, swap $A[p]$ and $A[\ell]$
$i \leftarrow \ell, j \leftarrow r$
while true **do**
    while $i < j$ and $A[i] < A[j]$ do $j \leftarrow j - 1$
    if $i = j$ **then** break
    swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$
    while $i < j$ and $A[i] < A[j]$ **do** $i \leftarrow i + 1$
    if $i = j$ **then** break
    swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$
return i

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

1: **if** $\ell \geq r$ **then return**
2: $m \leftarrow$ patition(A, ℓ, r)
3: quicksort$(A, \ell, m-1)$
4: quicksort$(A, m+1, r)$

- To sort an array A of size n, call quicksort$(A, 1, n)$.

Note: We pass the array A by reference, instead of by copying.

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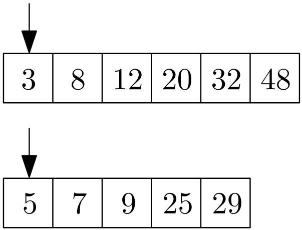

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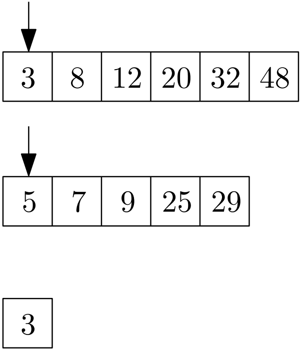

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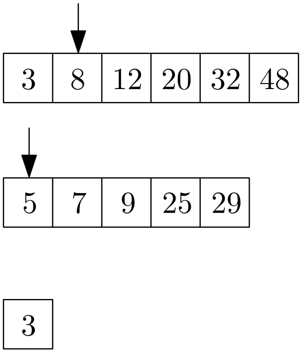

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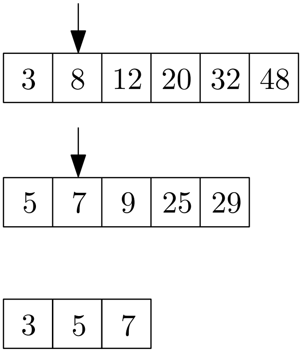

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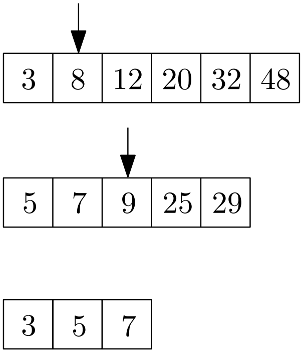

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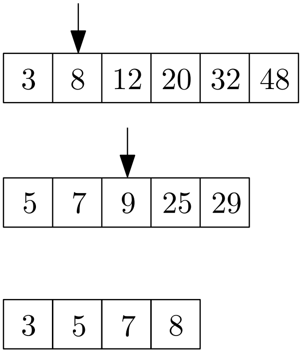

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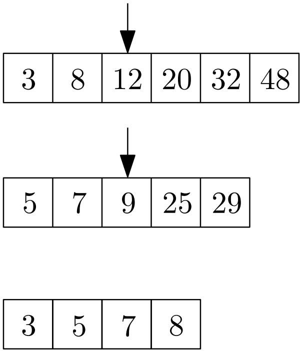

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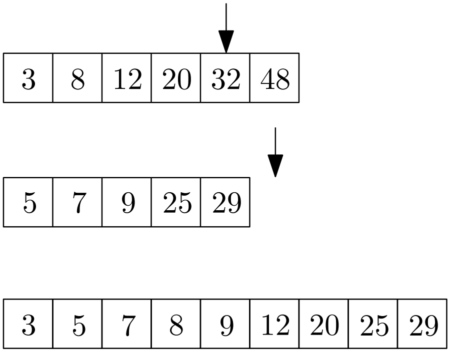

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