Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

Two Methods to Build a MST

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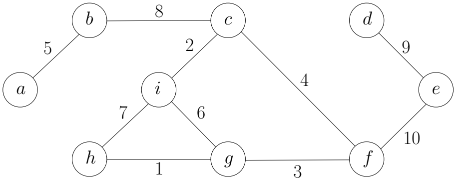

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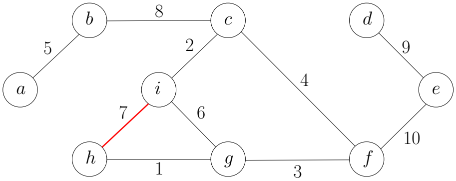

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A: The heaviest non-bridge edge.

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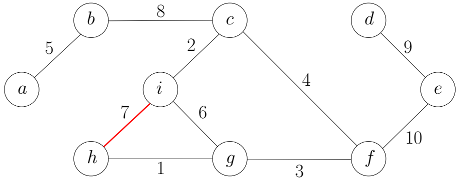


Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

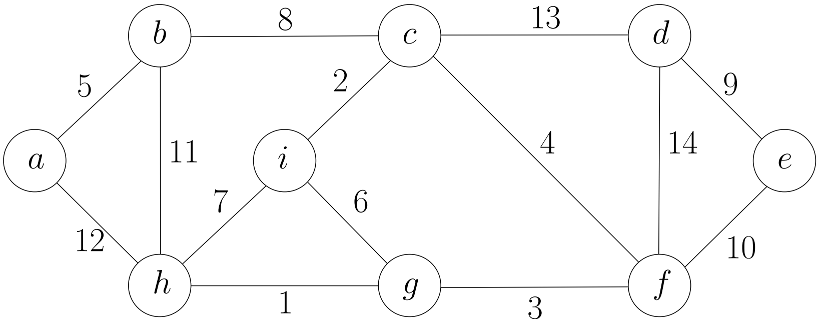
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

```
F ← E
sort E in non-increasing order of weights
for every e in this order do
    if (V, F \ {e}) is connected then
        F ← F \ {e}
return (V, F)
```

Reverse Kruskal's Algorithm: Example



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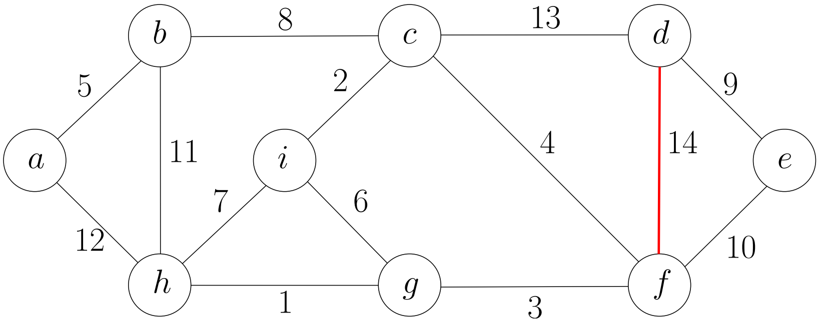

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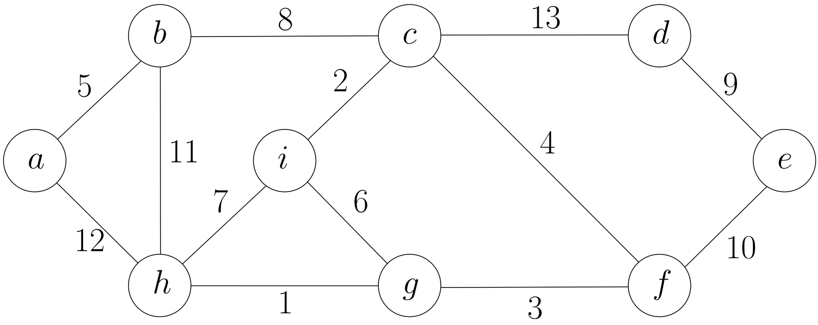

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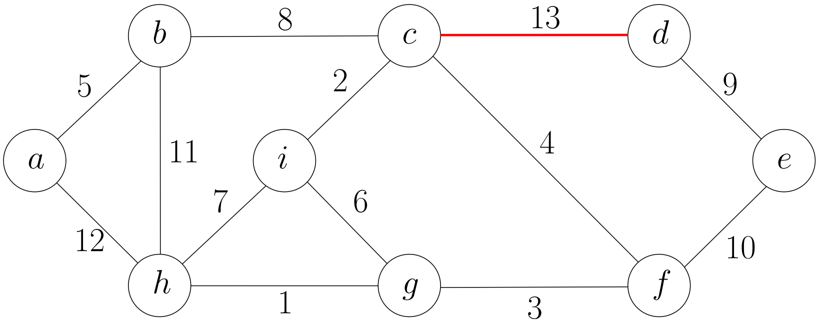

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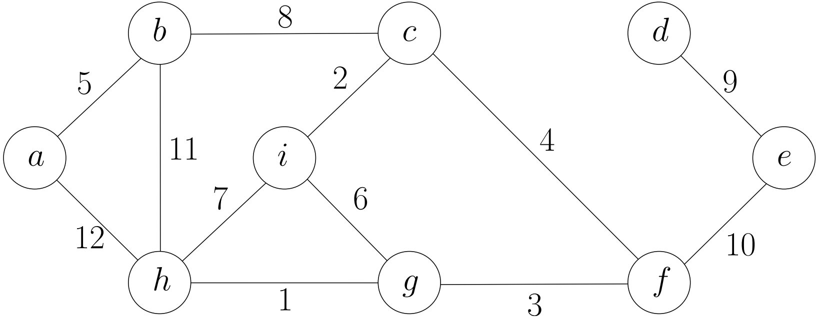

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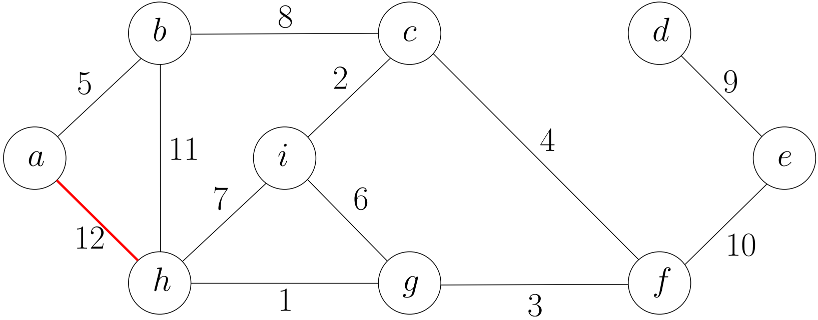

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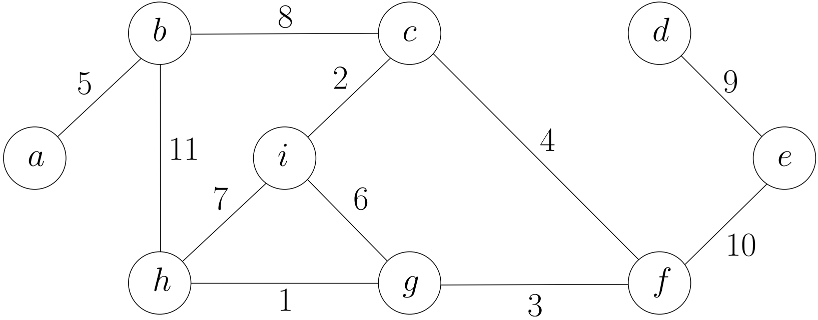

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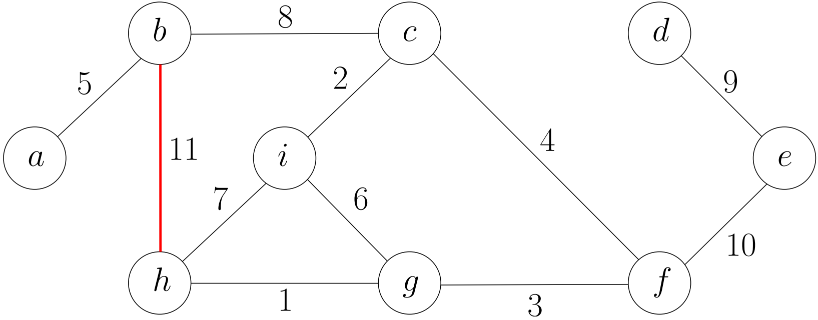

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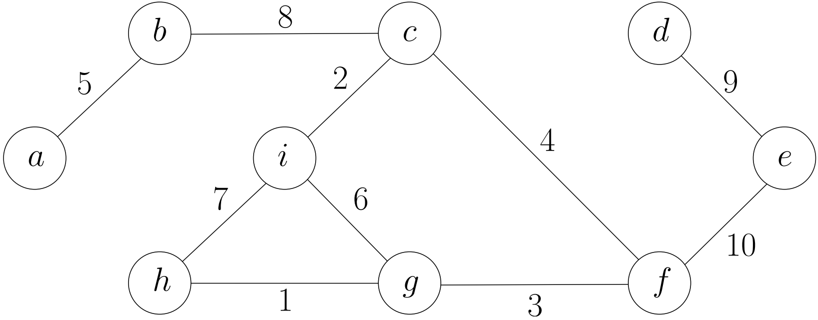

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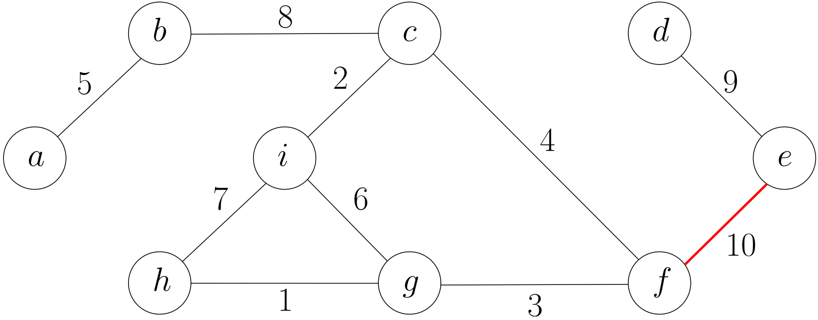

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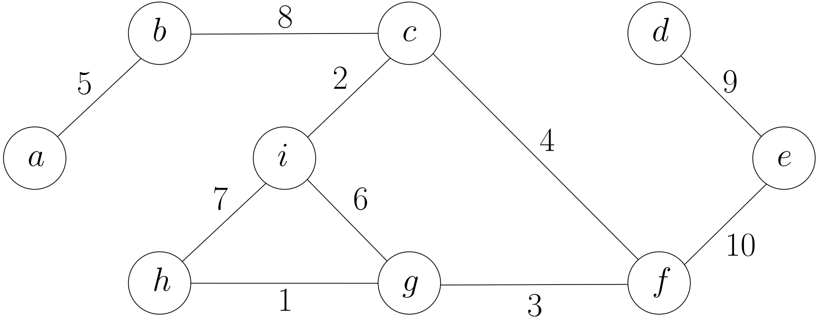

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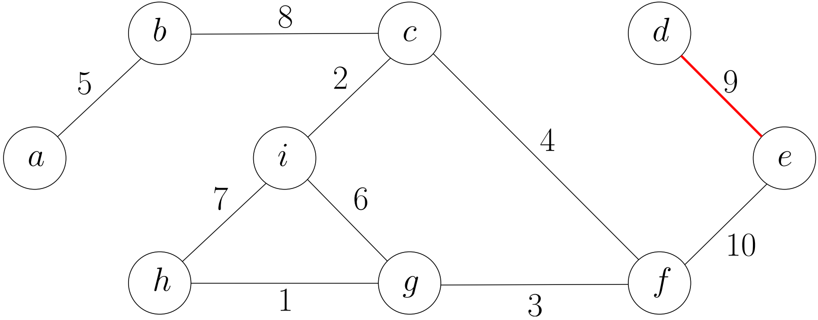

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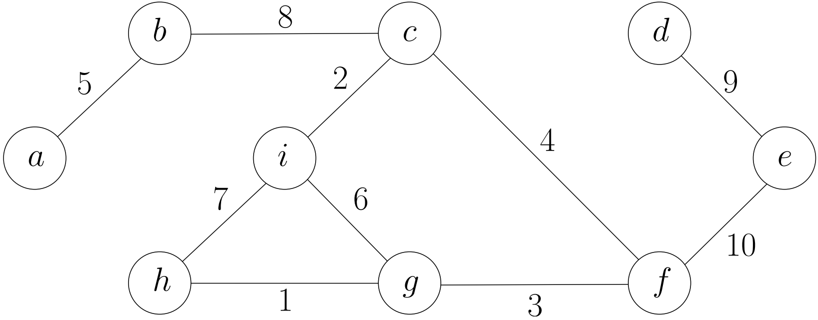

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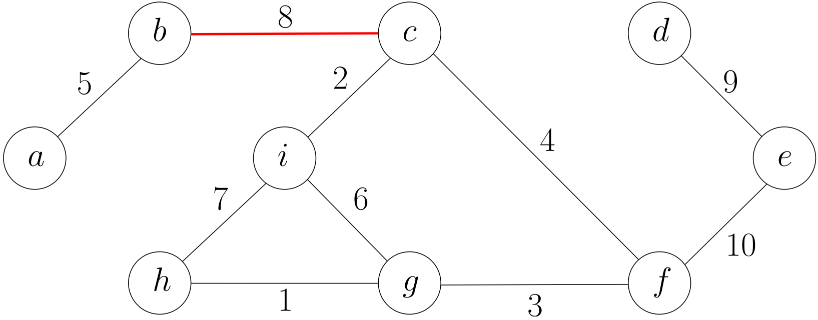

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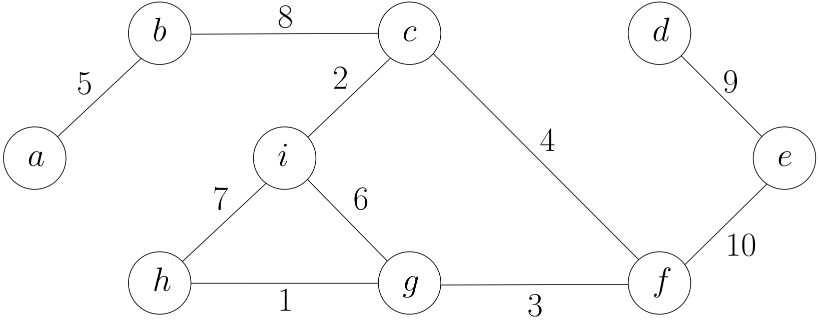

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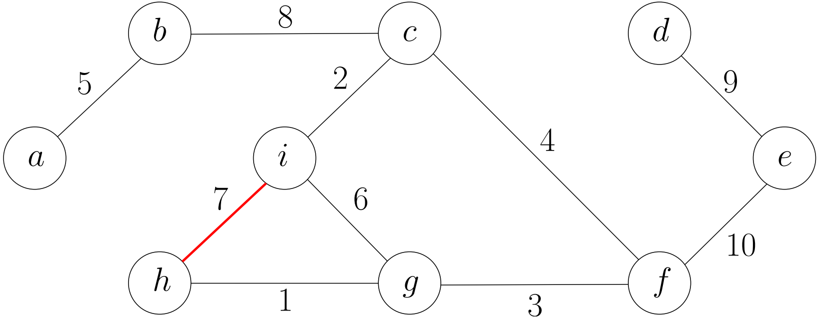

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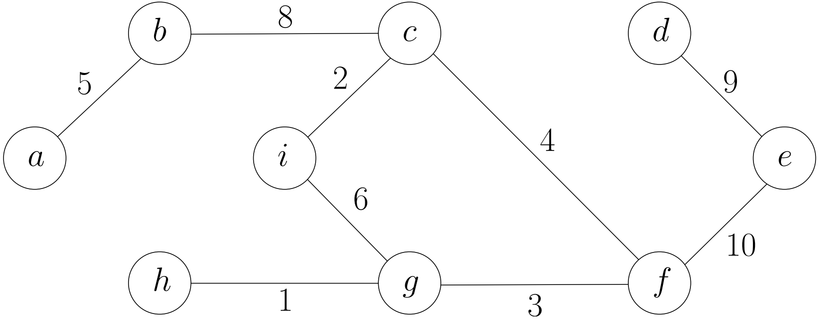

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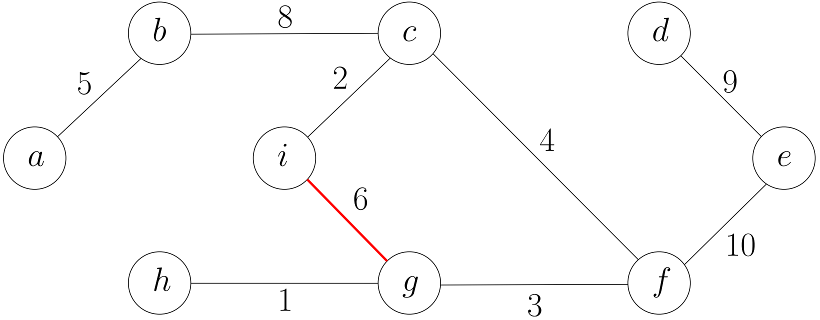

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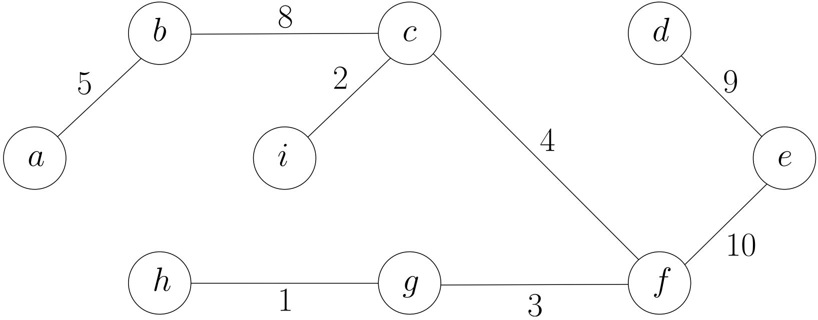

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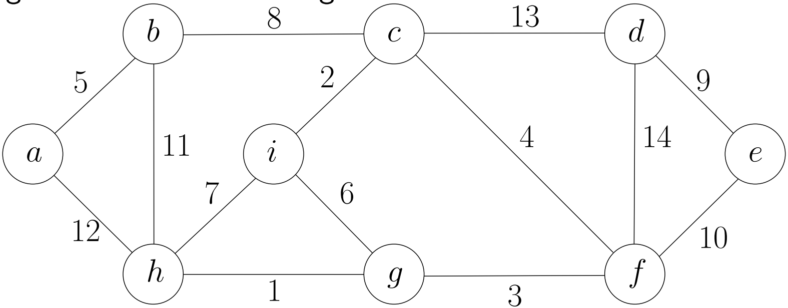
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



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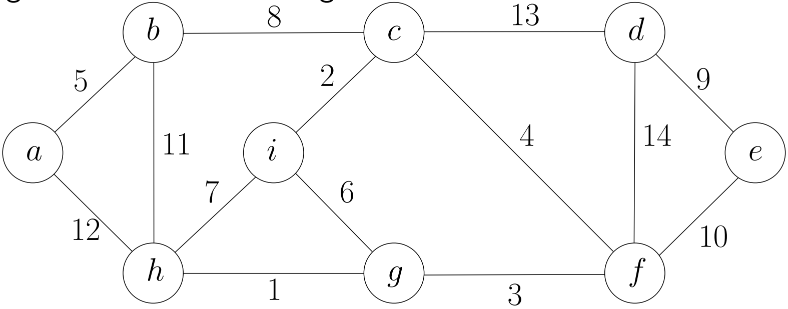

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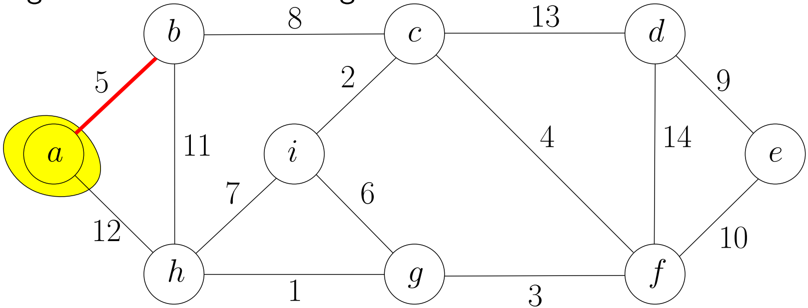

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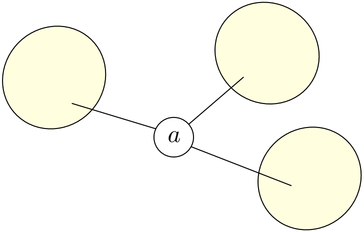

Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

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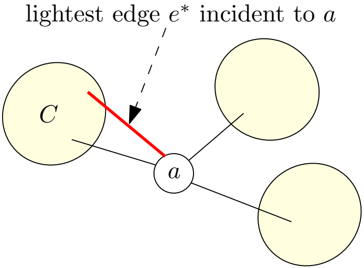

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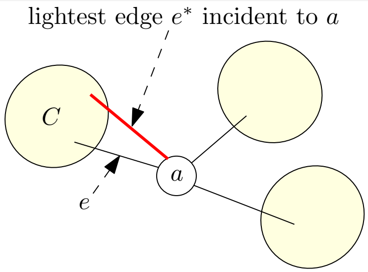

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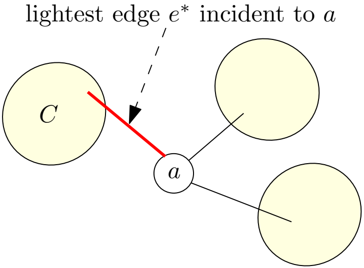

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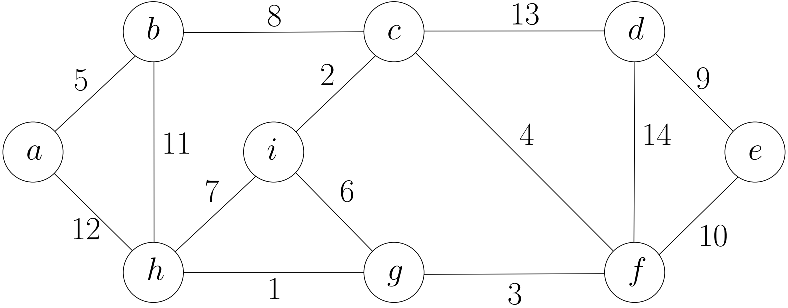
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ $\square$

Prim's Algorithm: Example



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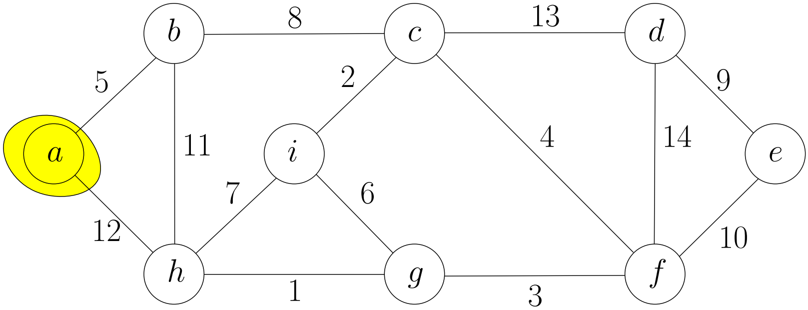

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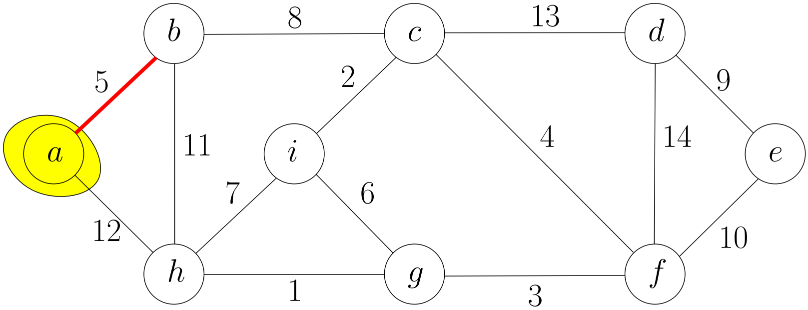

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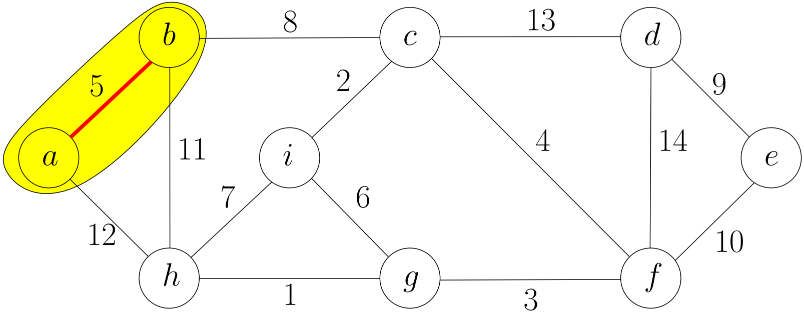

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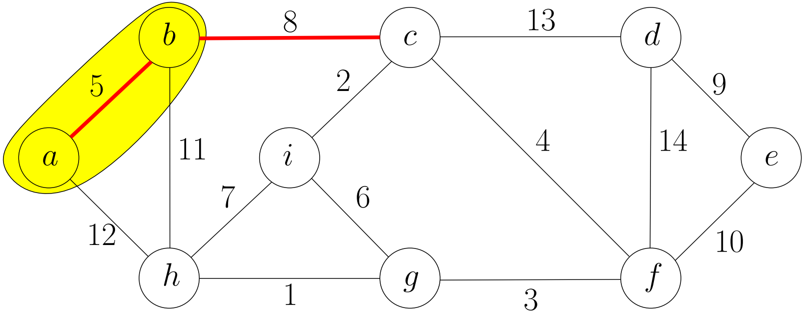

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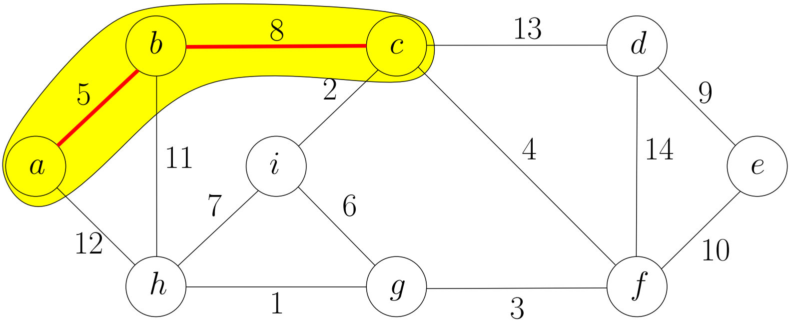

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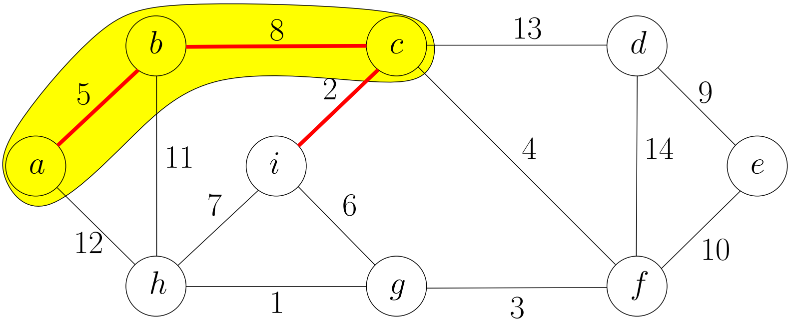

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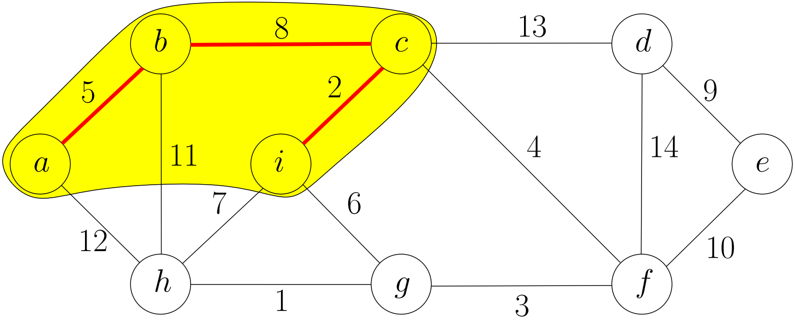

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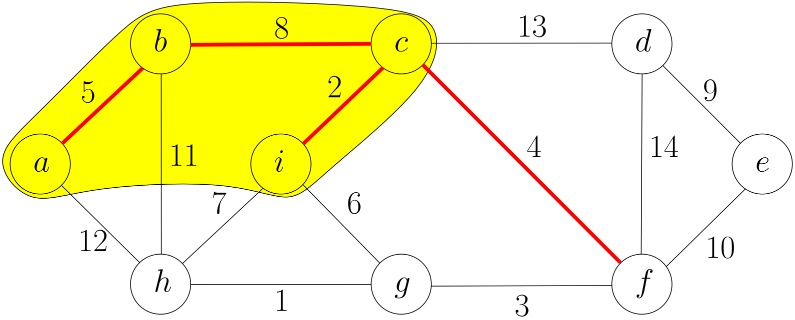

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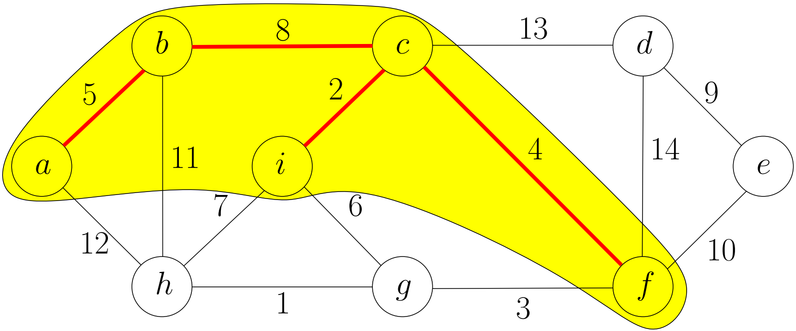

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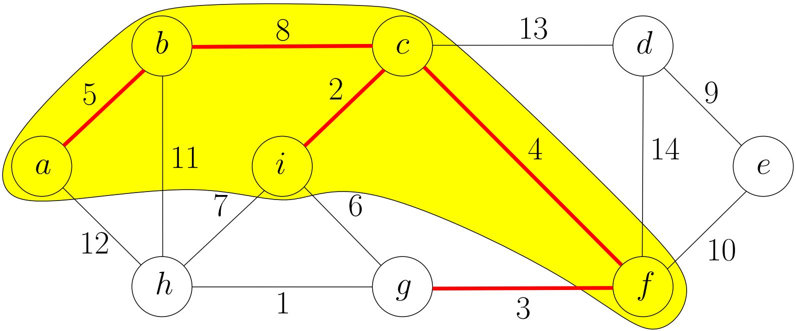

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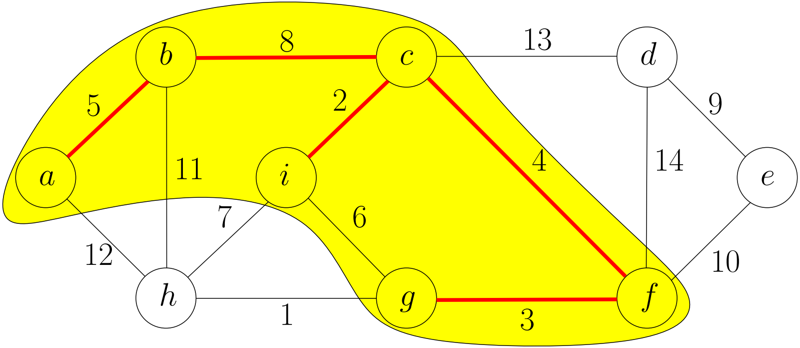

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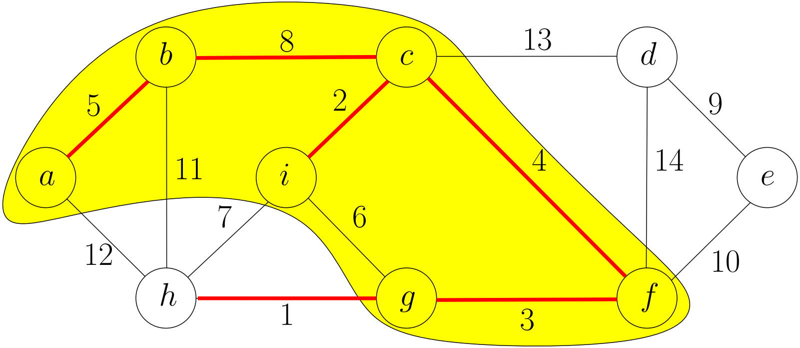

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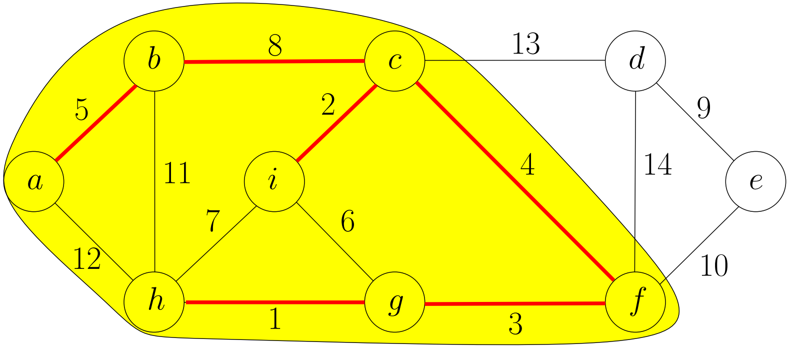

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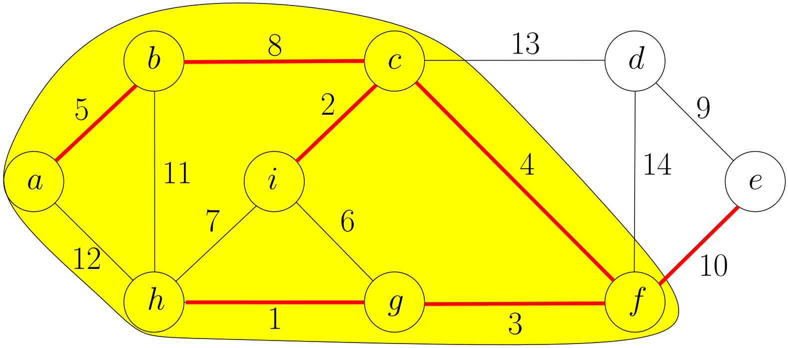

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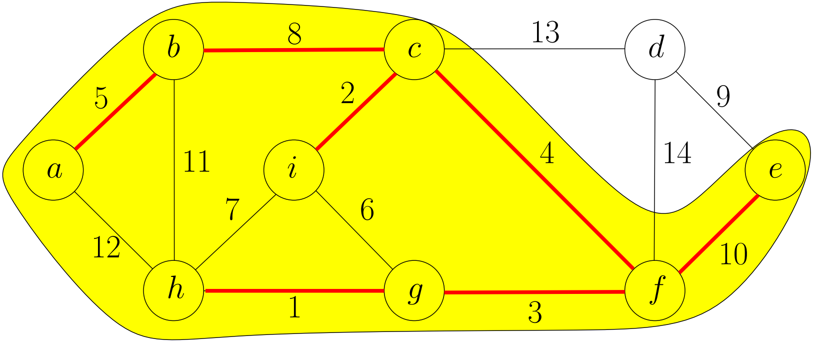

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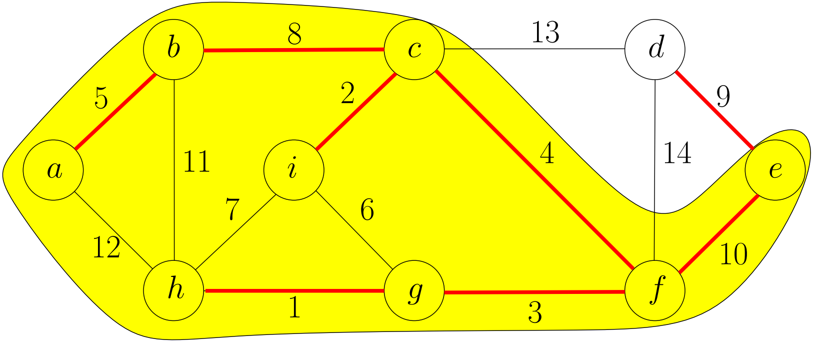

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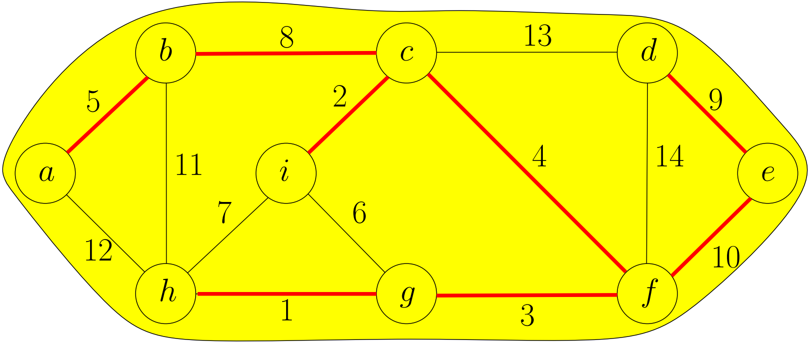


Greedy Algorithm

MST-Greedy1(G, w)

$S \leftarrow \{s\}$, where s is arbitrary vertex in V
$F \leftarrow \emptyset$
while $S \neq V$ **do**
    $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
        where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

Greedy Algorithm

MST-Greedy1(G, w)

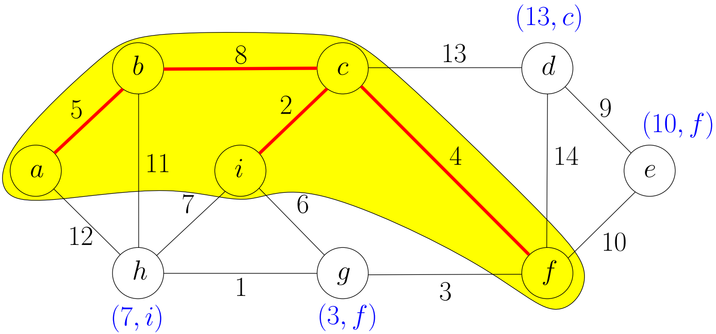
```
S ← {s}, where s is arbitrary vertex in V
F ← ∅
while S ≠ V do
    (u, v) ← lightest edge between S and V \ S,
                         where u ∈ S and v ∈ V \ S
    S ← S ∪ {v}
    F ← F ∪ {(u, v)}
return (V, F)
```

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S



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- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u, v)$:
  $(\pi[v], v)$ is the lightest edge between v and S

In every iteration

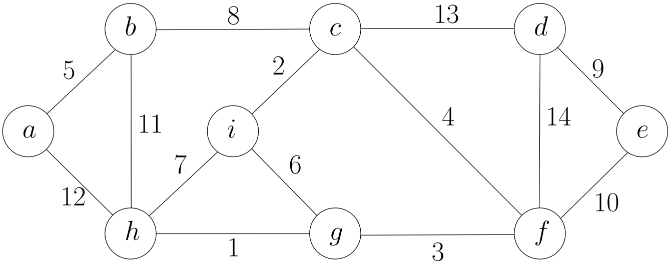
- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
while S ≠ V do
    u ← vertex in V \ S with the minimum d[u]
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if w(u, v) < d[v] then
            d[v] ← w(u, v)
            π[v] ← u
return {(u, π[u]) | u ∈ V \ {s}}
```

Example



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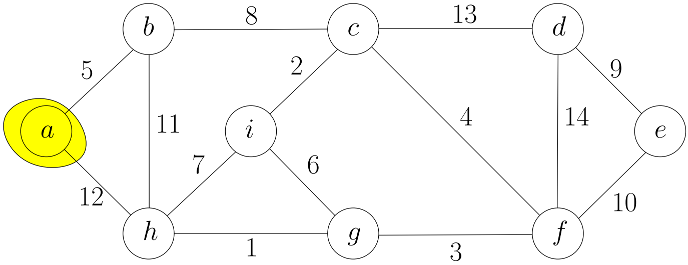

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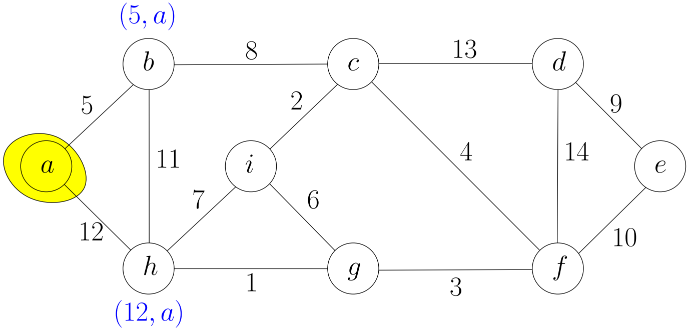

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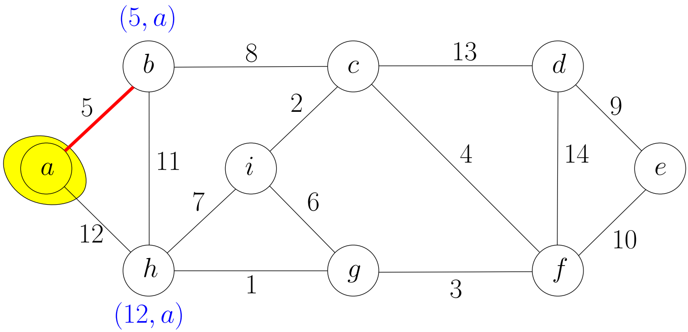

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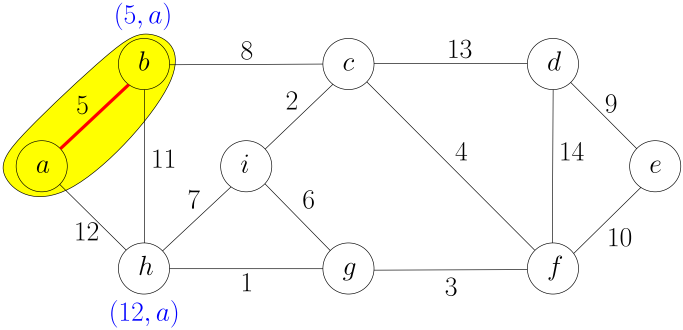

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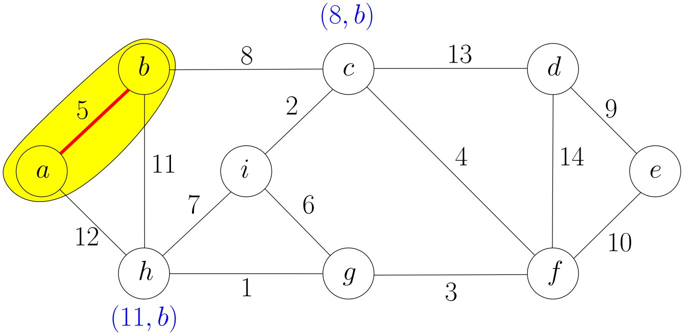

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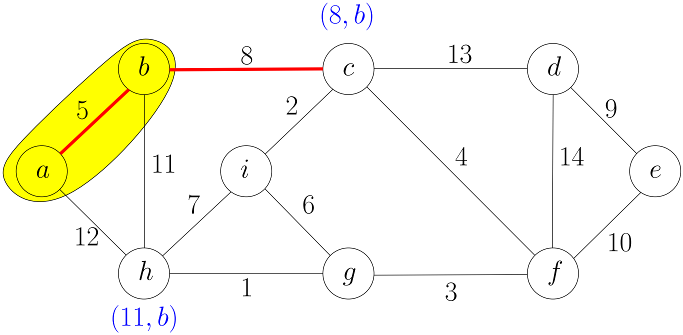

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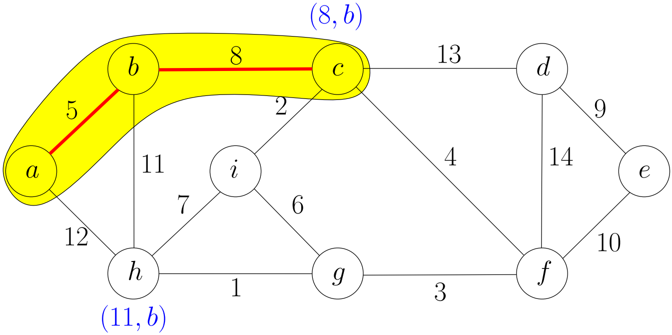

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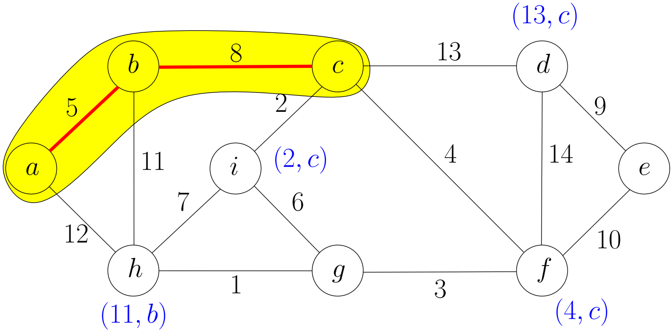
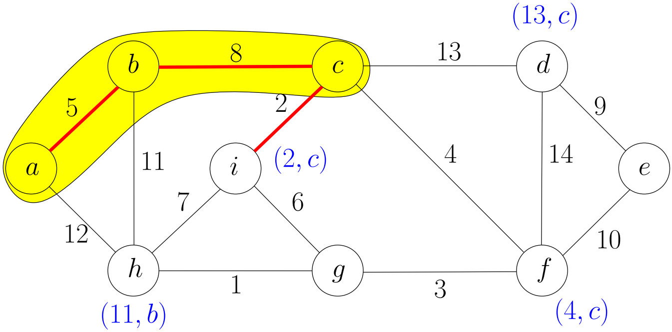


Example



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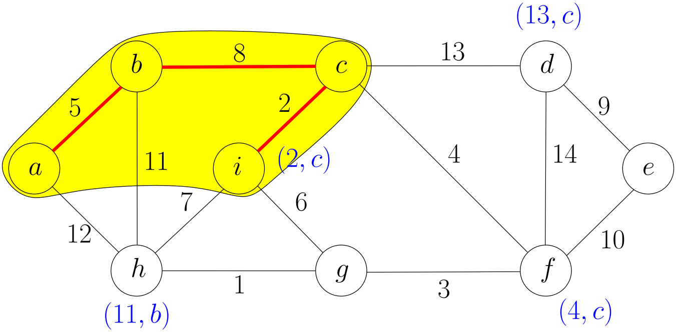

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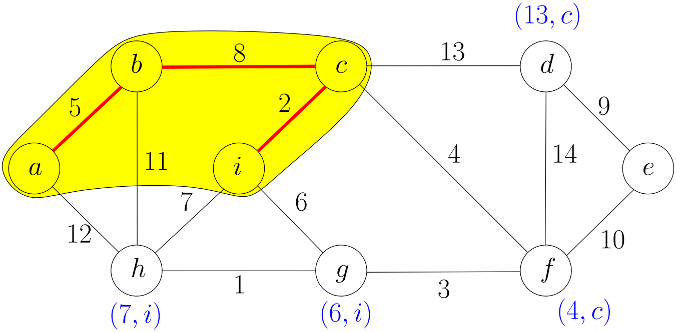

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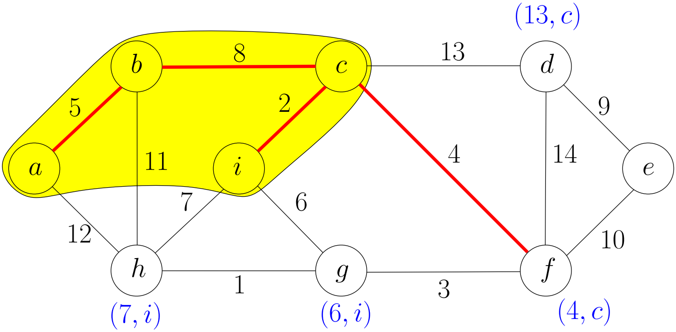

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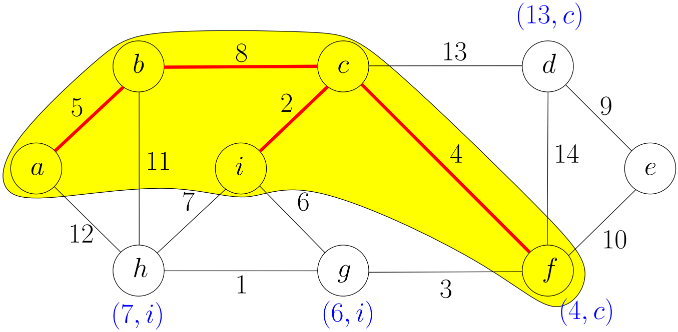

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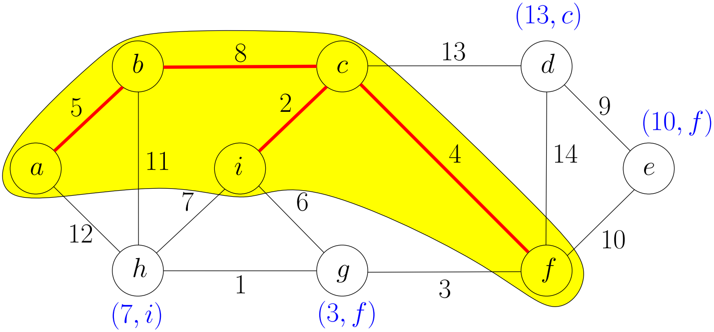

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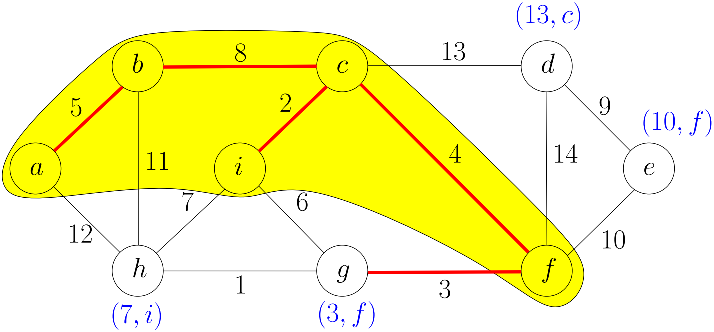

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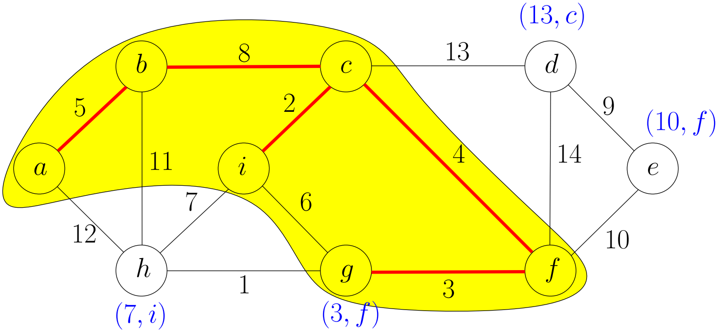

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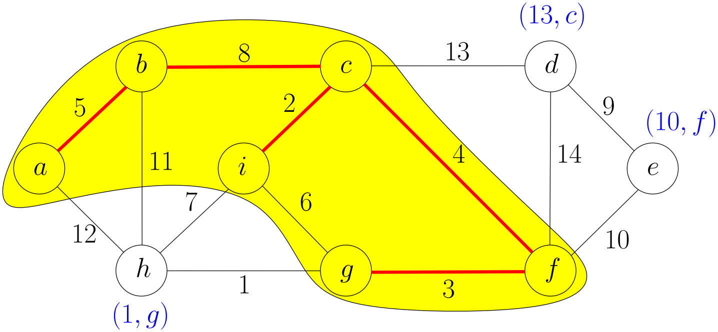

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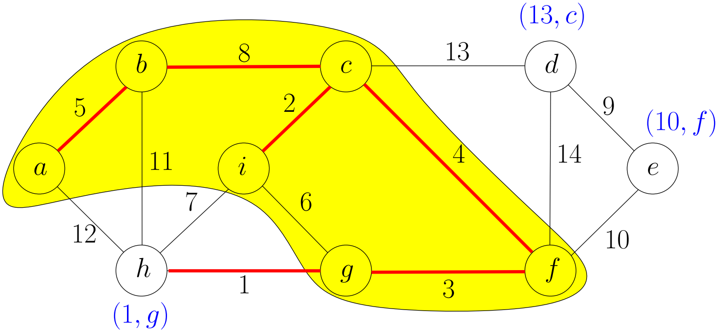

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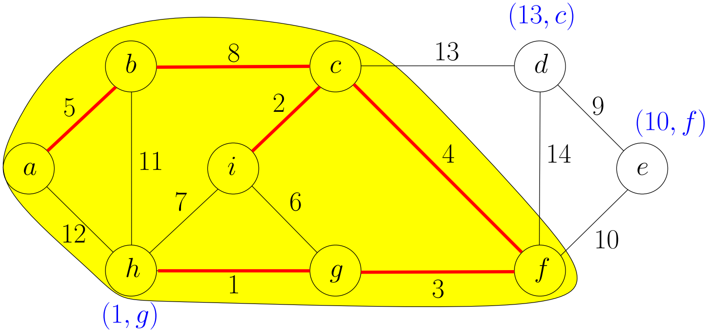

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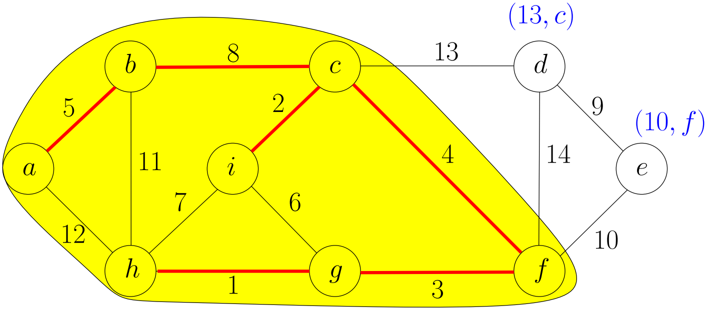

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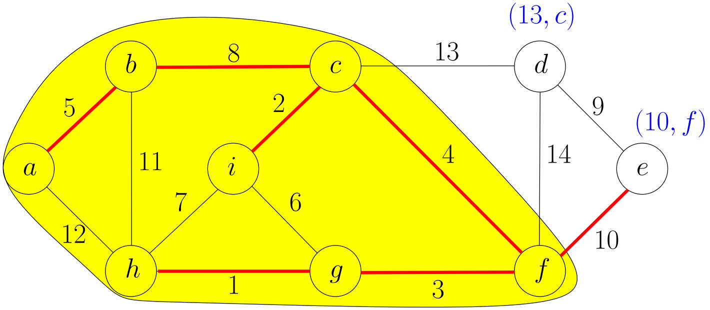

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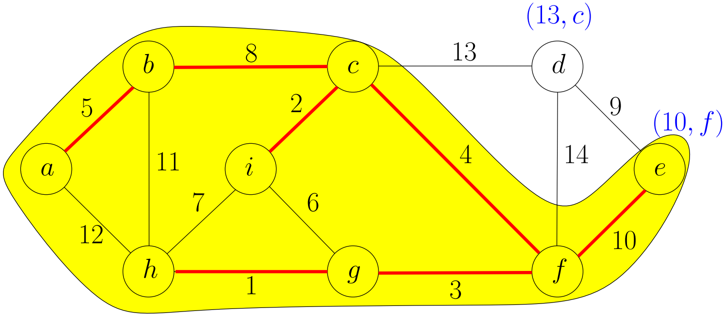

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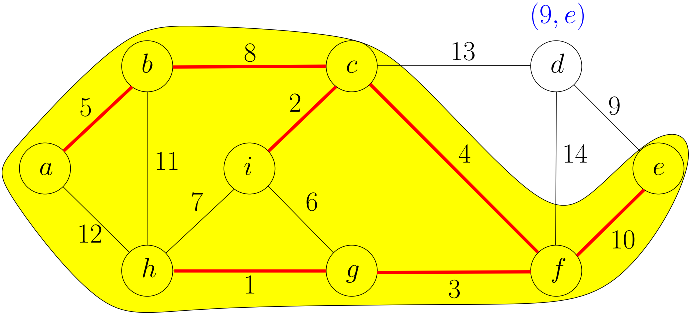

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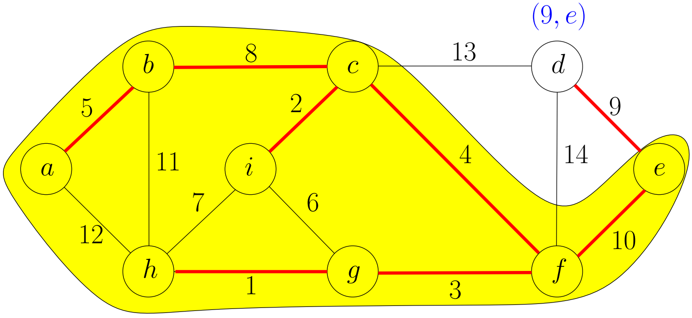

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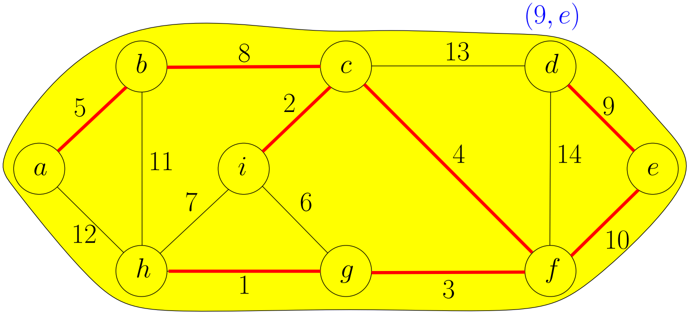

Example