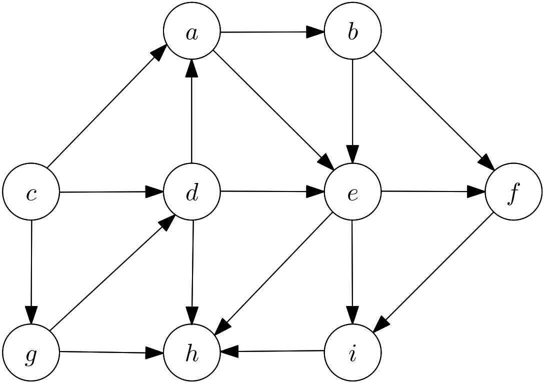
Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



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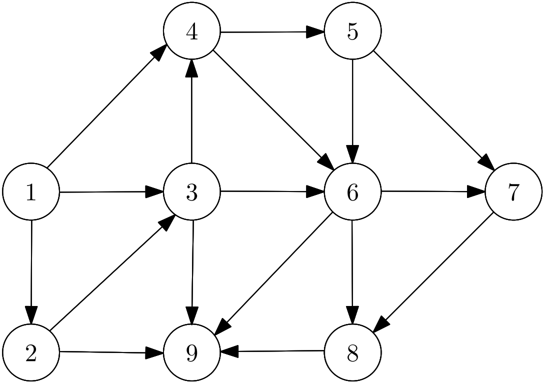
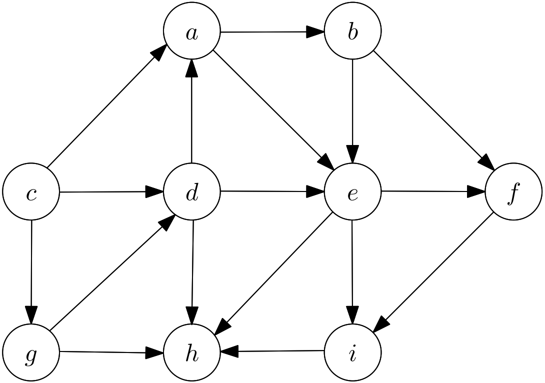


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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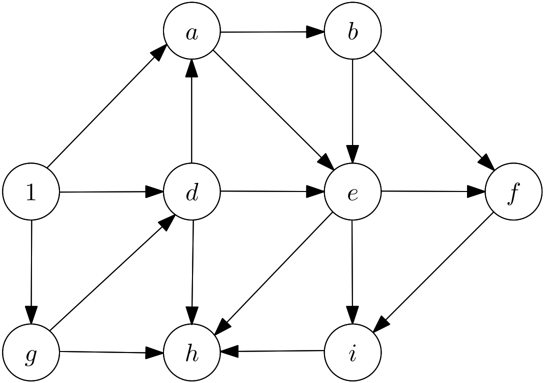

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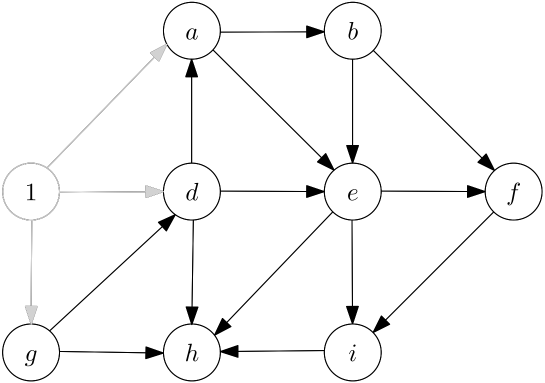

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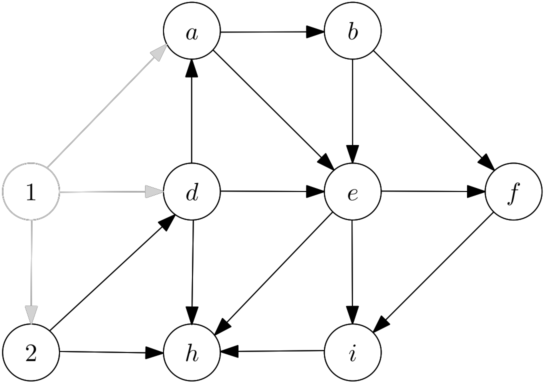

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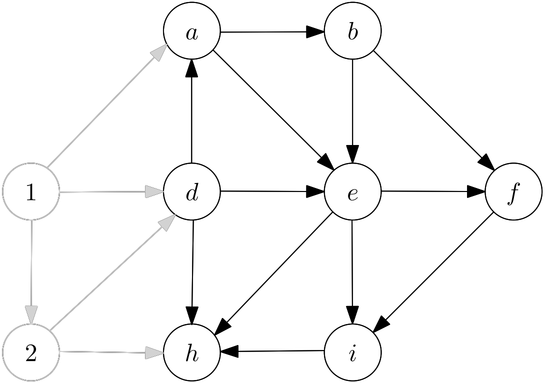

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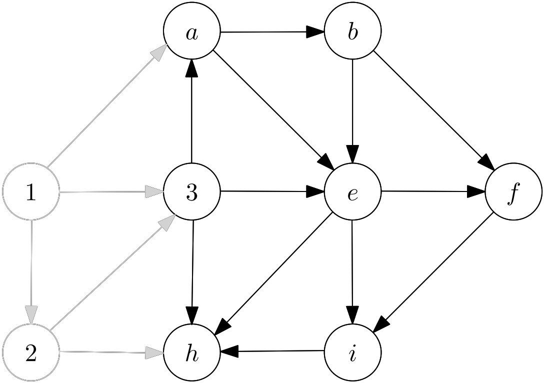

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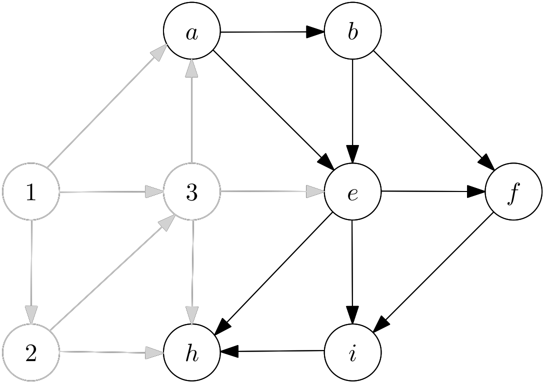

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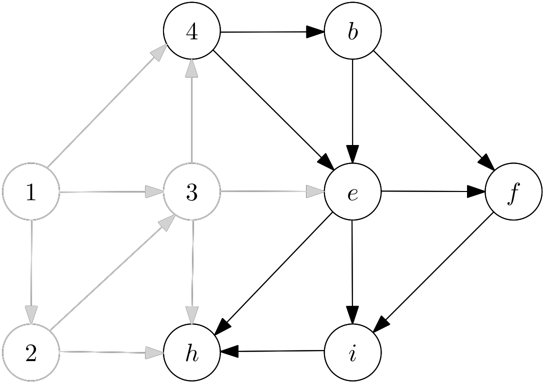

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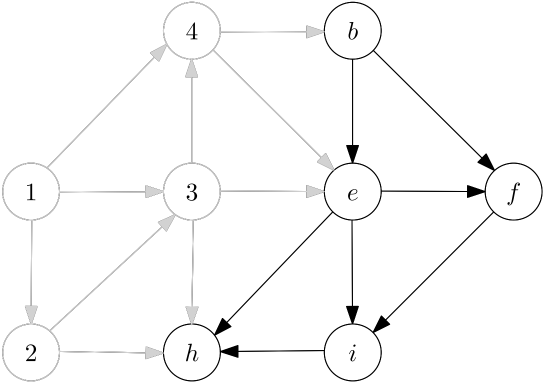

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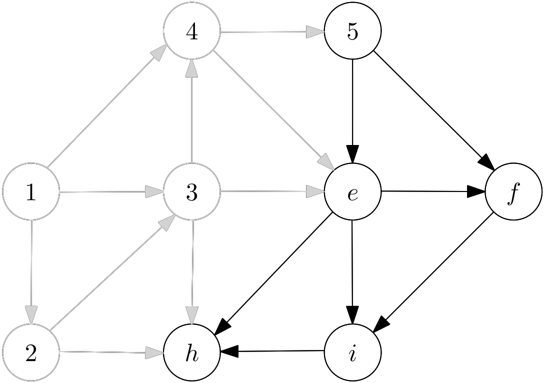

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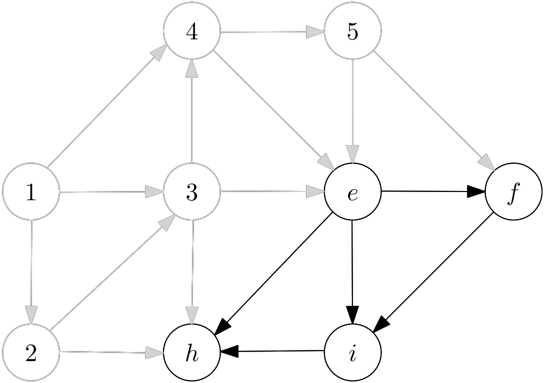

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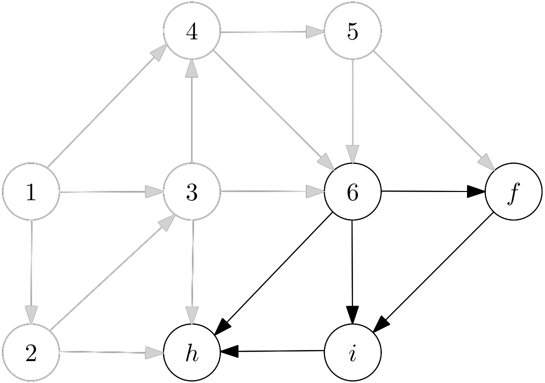

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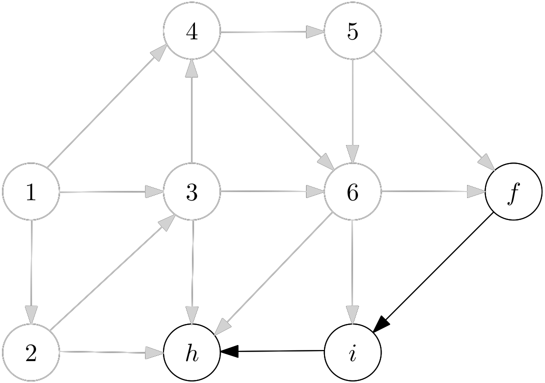

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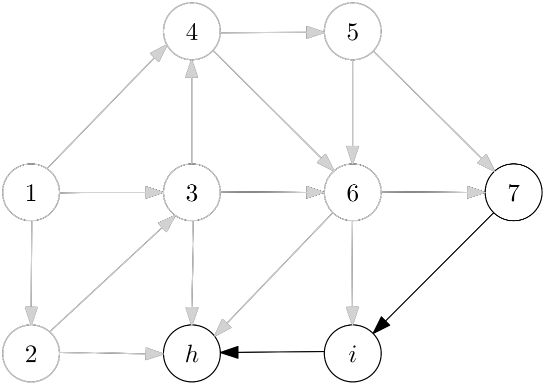

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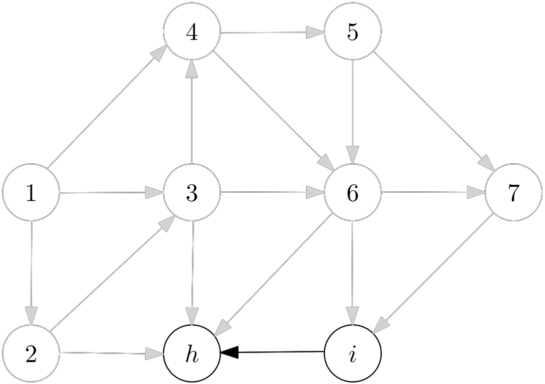

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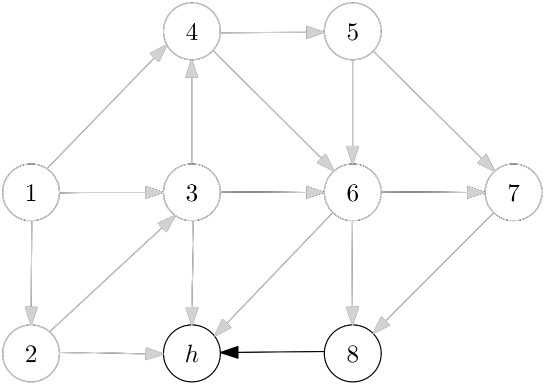

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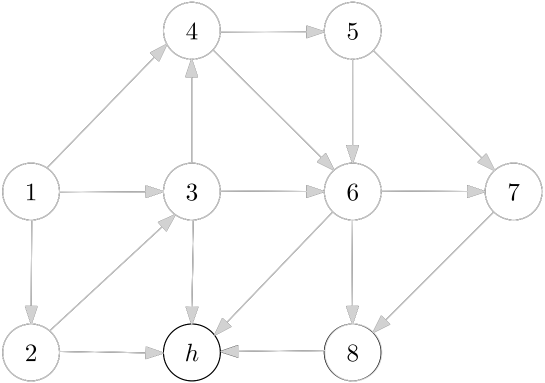

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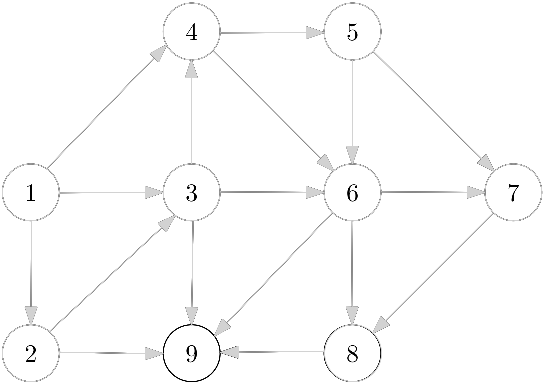

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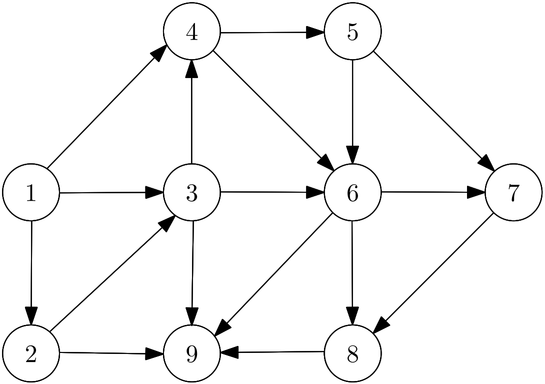

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Q: How to make the algorithm as efficient as possible?

Topological Ordering

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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

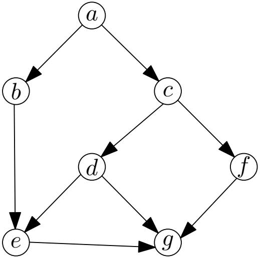
```
let $d_v \leftarrow 0$ for every $v \in V$
for every $v \in V$ do
    for every $u$ such that $(v, u) \in E$ do
        $d_u \leftarrow d_u + 1$
$S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
while $S \neq \emptyset$ do
    $v \leftarrow$ arbitrary vertex in $S$, $S \leftarrow S \setminus \{v\}$
    $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
    for every $u$ such that $(v, u) \in E$ do
        $d_u \leftarrow d_u - 1$
        if $d_u = 0$ then add $u$ to $S$
if $i < n$ then output “not a DAG”
```

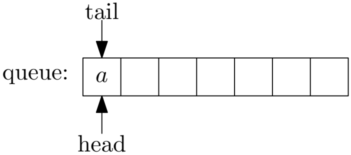
- S can be represented using a queue or a stack
- Running time $= O(n + m)$

S as a Queue or a Stack

DS	Queue	Stack
Initialization	$head \leftarrow 1,\ tail \leftarrow 0$	$top \leftarrow 0$
Non-Empty?	$head \leq tail$	$top > 0$
Add(v)	$tail \leftarrow tail + 1$ $S[tail] \leftarrow v$	$top \leftarrow top + 1$ $S[top] \leftarrow v$
Retrieve v	$v \leftarrow S[head]$ $head \leftarrow head + 1$	$v \leftarrow S[top]$ $top \leftarrow top - 1$

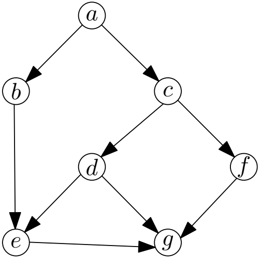
Example

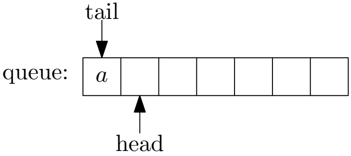




	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

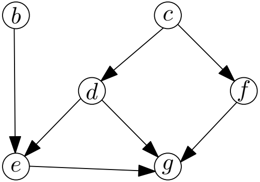
Example

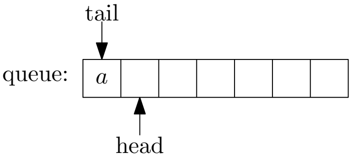




	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

Example





	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

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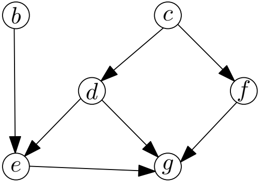

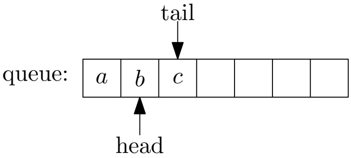

	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

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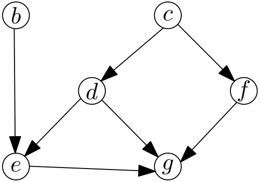
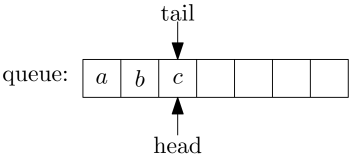
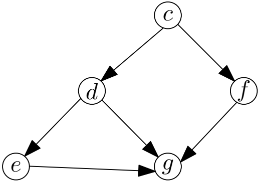
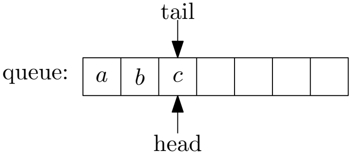




	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

Example





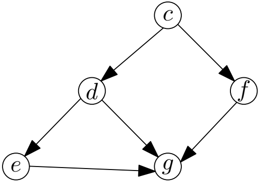
tail

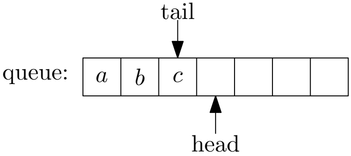
queue:	a	b	c				

head

	a	b	c	d	e	f	g
degree	0	0	0	1	1	1	3

Example





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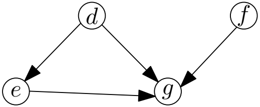
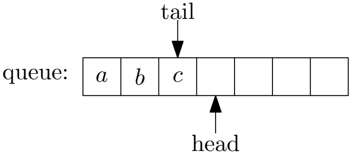
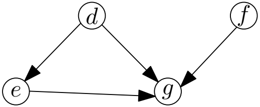
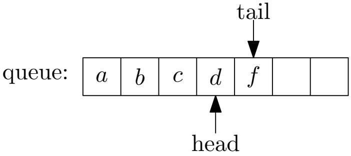




	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

Example





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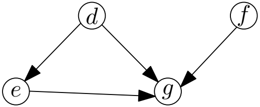
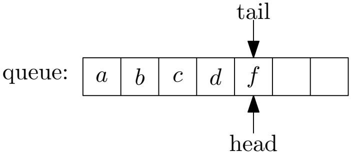
Example





tail

queue:

a	b	c	d	f		

head

	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

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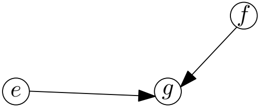

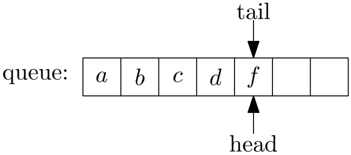

tail

queue:

a	b	c	d	f		

head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

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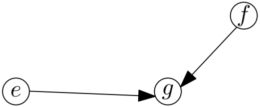
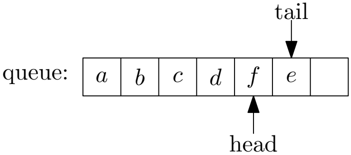
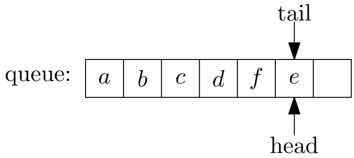




	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

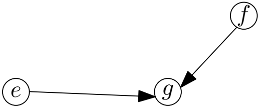
Example



tail

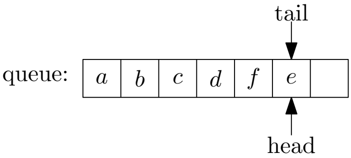
queue:	a	b	c	d	f	e	

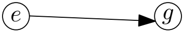
head



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

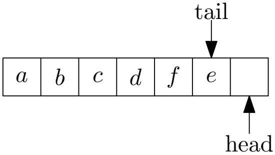
Example





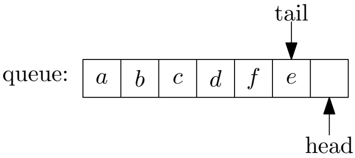
	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example



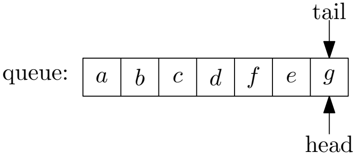
	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example



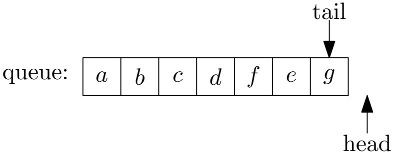
	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Example



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Example



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Outline

1. Graphs
2. Connectivity and Graph Traversal
    - Types of Graphs
3. Bipartite Graphs
    - Testing Bipartiteness
4. Topological Ordering
    - Applications: Word Ladder

Def. Word: A string formed by letters.

Def. Adjacency words: Word A and B are adjacent if they differ in exactly one letter.

e.g. wor$d$ and wor$k$; t$e$ll and t$a$ll; ask$b$e and ask$e$e.

Def. Word Ladder: Players start with one word and, in a series of steps, change or transform that word into another word.

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- The objective is to make the change in the smallest number of steps, with each step involving changing a **single letter** of the word to create a new valid word.

Word Ladder Problem

Input: Two words S and T, a list of words $A = \{W_1, W_2, ..., W_k\}$.

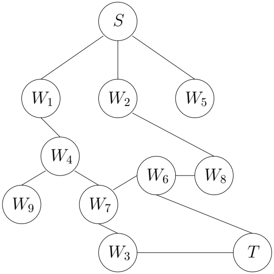
Output: " The smallest word ladder" if we can change S to T by moving between adjacency words in $A \cup \{S, T\}$; Otherwise, "No word ladder".

Example:

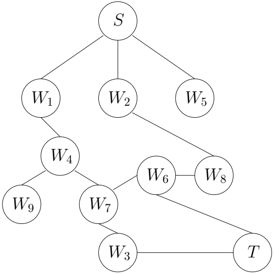
- S="a e f g h", T = "d l m i h"
- W_1="a e f i h", W_2 = "a e m g h", W_3="d l f i h"
  W_4 = "s e f i h", W_5="a d f g h", W_6 = "d e m i h"
  W_7="d e f i h", W_8 = "d e m g h", W_9 = "s e m i h"

Example:

- S="a e f g h", T = "d l m i h"
- W_1="a e f i h", W_2 = "a e m g h", W_3="d l f i h"
  W_4 = "s e f i h", W_5="a d f g h", W_6 = "d e m i h"
  W_7="d e f i h", W_8 = "d e m g h", W_9 = "s e m i h"



- Each vertex corresponds to a word.
- Two vertices are adjacent if the corresponding words are adjacent.



- Each vertex corresponds to a word.
- Two vertices are adjacent if the corresponding words are adjacent.
- Hints: Given vertex v, check its nearest neighbor.

CSE 431/531B: Algorithm Analysis and Design (Fall 2023)

Greedy Algorithms

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University at Buffalo

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Goals of algorithm design

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems

Common Paradigms for Algorithm Design

- Greedy Algorithms: shortest path problem
- Divide and Conquer: merge-sort, binary search
- Dynamic Programming: Fibonacci number

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- They often run in polynomial time due to their simplicity: easy to come up with, easy to analyze running time.
- Hard to see correctness. Mostly, it is not correct. E.g. $\min f(x)$

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- Safety: Prove that the reasonable strategy is "safe"
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

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Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.

Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
    - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

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Output: A way to put as many items as possible in the boxes.

Example:

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: $45 \to 60, 20 \to 40, 19 \to 25$

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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?