

Combinatorial Topology for Meshing

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Appel Seminar
March 8, 2022



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Exascale Simulation Technology



There is an isomorphism
between meshes and DAGs.

Outline

Meshes

Mesh Transformations

Future Work

Hasse Diagram

Mesh Representation (**KnepleyKarpeev09**;
LangeMitchellKnepleyGorman2015)
for a CW-complex (**Whitehead1949**)

DAG vertices $\longleftrightarrow$ k -cells

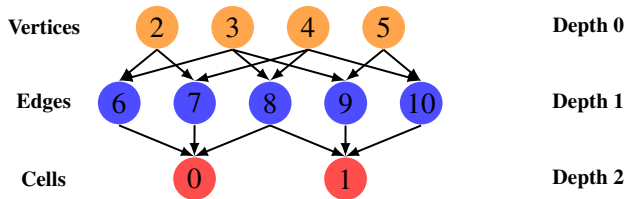
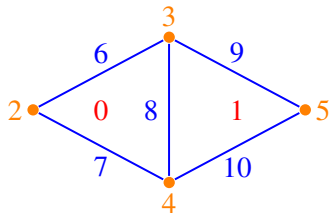
DAG edges $\longleftrightarrow$ boundary adjacency

Edge labels $\longleftrightarrow$ dihedral group representers

We call each DAG vertex a *mesh point*

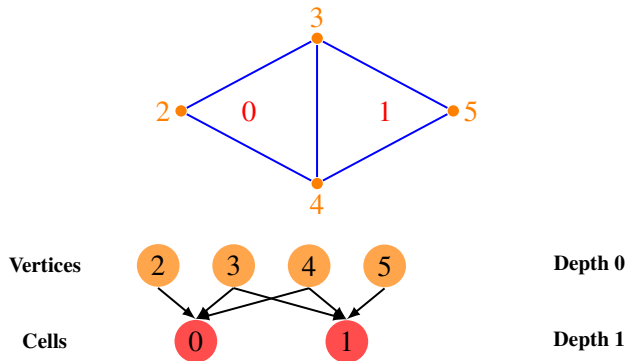
Sample Meshes

Interpolated triangular mesh



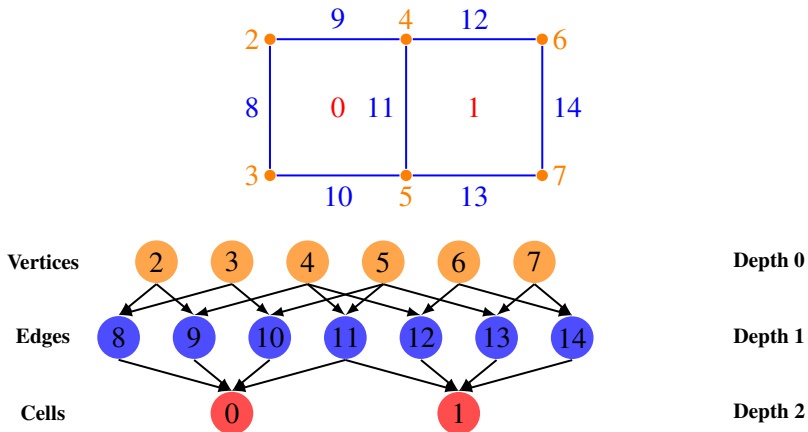
Sample Meshes

Optimized triangular mesh



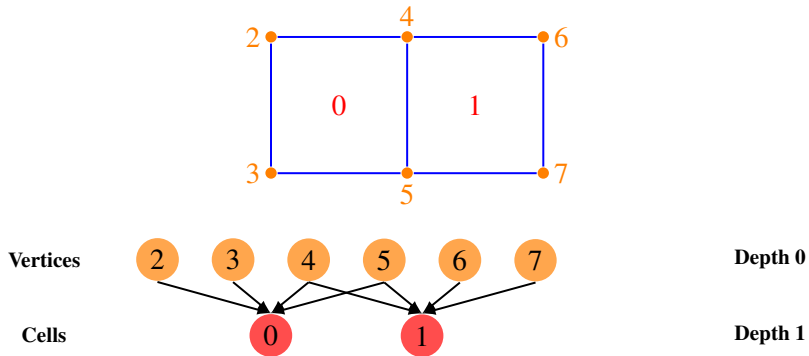
Sample Meshes

Interpolated quadrilateral mesh



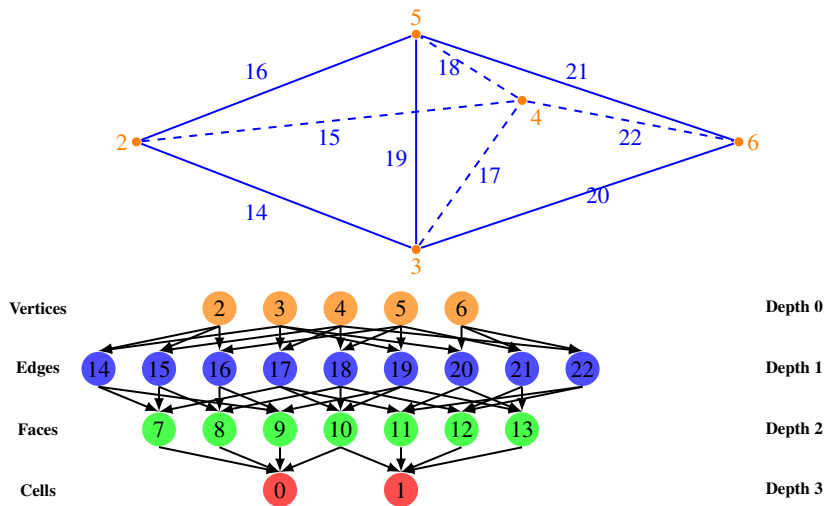
Sample Meshes

Optimized quadrilateral mesh



Sample Meshes

Interpolated tetrahedral mesh

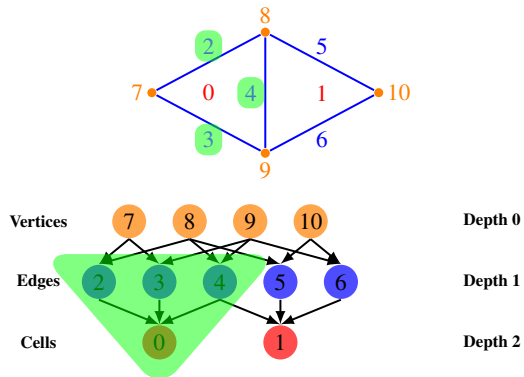


Basic Operations

Cone

We begin with the basic covering relation,

$$\text{cone}(0) = \{2, 3, 4\}$$

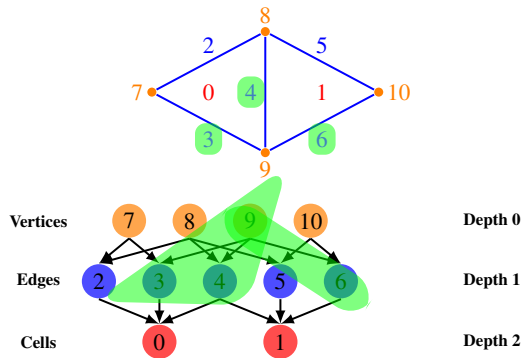


Basic Operations

Support

reverse arrows to get the
dual operation,

$\text{support}(9) = \{3, 4, 6\}$

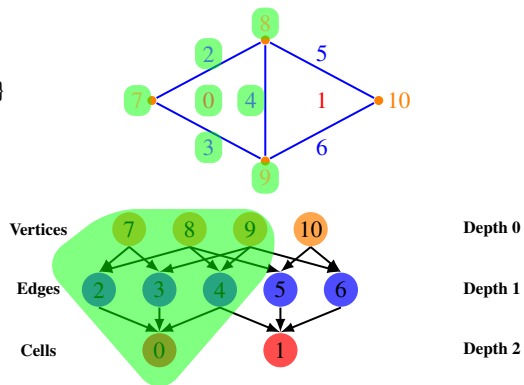


Basic Operations

Closure

add the transitive closure of
the relation,

$$\text{closure}(0) = \{0, 2, 3, 4, 7, 8, 9\}$$

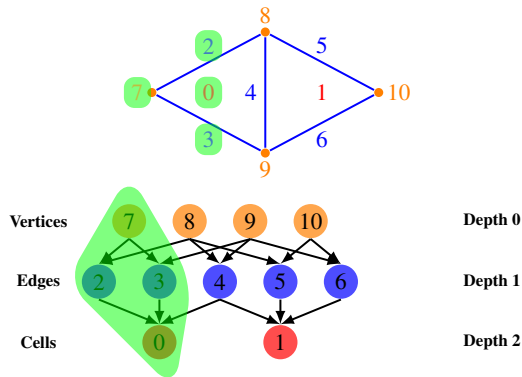


Basic Operations

Star

and the transitive closure of
the dual,

$$\text{star}(7) = \{7, 2, 3, 0\}$$

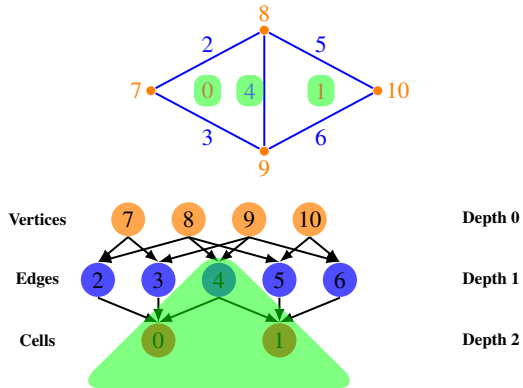


Basic Operations

Meet

and augment with lattice operations.

$$\text{meet}(0, 1) = \{4\}$$

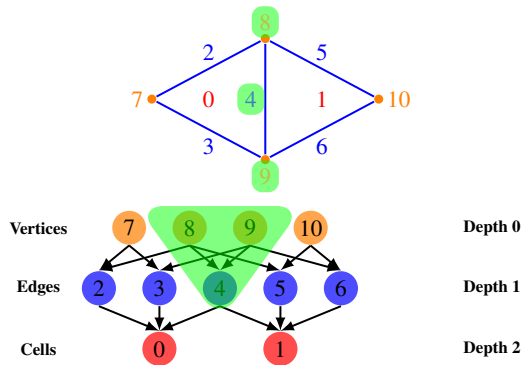


Basic Operations

Join

and augment with lattice operations.

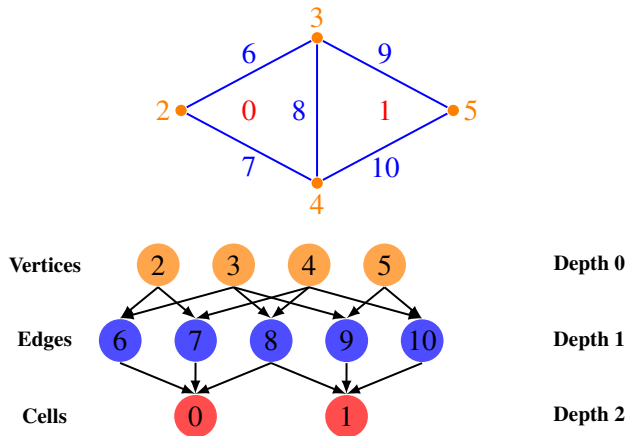
$$\text{join}(8, 9) = \{4\}$$



Sample Meshes

Interpolated triangular mesh

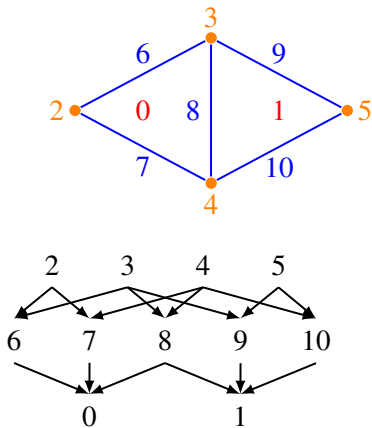
Starting with an interpolated mesh,



Sample Meshes

Interpolated triangular mesh

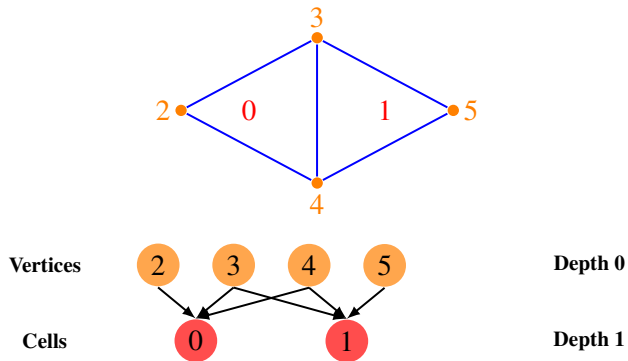
we can infer the depth in $\mathcal{O}(N)$ (BFS),



Sample Meshes

Interpolated triangular mesh

and we can infer the intermediate levels in $\mathcal{O}(N)$ (Join).



Cell Orientation

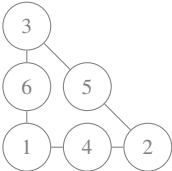
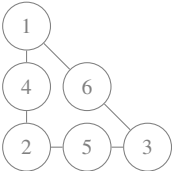
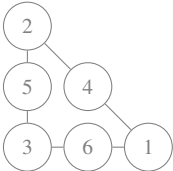
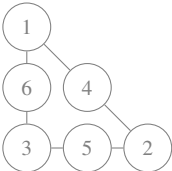
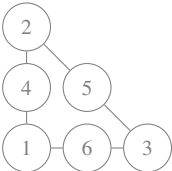
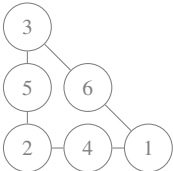
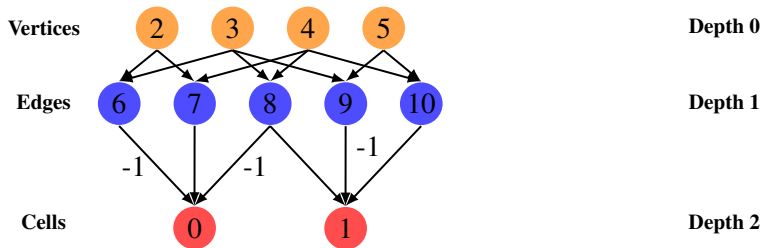
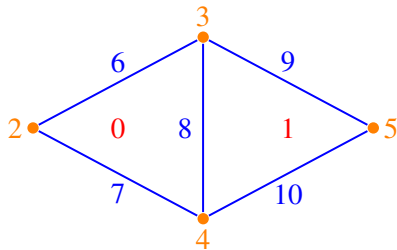
Orientation	0	1	2
Arrangement			
Orientation	-1	-2	-3
Arrangement			

Table: The dihedral group D_3 for the triangle

Orientation

Edge Decoration



Outline

Meshes

Mesh Transformations

Future Work

Transformation

Production Rule:

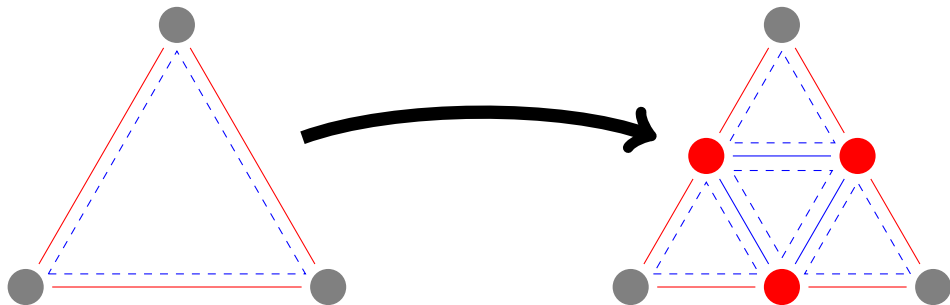
$$p \rightarrow \{(q_k, o_k)\}$$

or

$$q_k \in \text{child}(p)$$

Example Transformation

Refinement



Transformation Conditions

Condition 1:

$$\text{cone}(\text{child}(p)) \in \text{child}(\text{cl}(p)),$$

Condition 2:

$$\text{parent}(\text{st}(q)) \in \text{st}(\text{parent}(q))$$

Transformation Conditions

Condition 1:

$$\text{cone}(\text{child}(p)) \in \text{child}(\text{cl}(p)),$$

Condition 2:

$$|\text{parent}(q)| = 1$$

Closure Complexity

Consider a point q' in the cone of q , so that

$$\begin{aligned} q' &\in \text{cone}(q) \\ &\in \text{child}(\text{cl}(p)) \end{aligned}$$

by Condition 1, meaning

$$\exists p' \in \text{cl}(p), q' \in \text{child}(p').$$

We may take the cone of each side and use Condition 1 again,

$$\begin{aligned} \text{cone}(q') &\in \text{cone}(\text{child}(p')) \\ &\in \text{child}(\text{cl}(p')) \\ &\in \text{child}(\text{cl}(p)) \end{aligned}$$

Thus

$$\text{cl}(\text{child}(p)) \in \text{child}(\text{cl}(p)).$$

Support Complexity

Let p produce q , and consider a point q' in the star of q ,

$$q' \in \text{st}(q) \iff q \in \text{cl}(q')$$

Let q' be produced by a point p' ,

$$\begin{aligned} q' &\in \text{child}(p') \\ \text{cl}(q') &\in \text{child}(\text{cl}(p')) \\ q &\in \text{child}(\text{cl}(p')) \end{aligned}$$

Using Condition 2,

$$\begin{aligned} \text{parent}(q) &\in \text{cl}(p') \\ p &\in \text{cl}(p') \\ p' &\in \text{st}(p) \end{aligned}$$

and thus

$$\text{st}(\text{child}(p)) \in \text{child}(\text{st}(p)).$$

Numbering

With unique parents,
we can number everything *a priori*,
in parallel.

Output Sensitivity

To generate a closure in the output mesh,
we need only the producing point closure.

To generate a star in the output mesh,
we need only the producing point star.

Outline

Meshes

Mesh Transformations

Future Work

Possible Proofs

- ▶ Transformation always produces a valid local CW-complex
- ▶ Transformation always produces a valid parallel CW-complex
- ▶ Transformation is output sensitive
- ▶ Transformation produces the intended output
- ▶ Parallel numbering with relaxed parent condition

References I