

Ephemeral Meshes

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April 13, 2023



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Meshes should consist of
data *and* code.

Outline

Mesh Representation

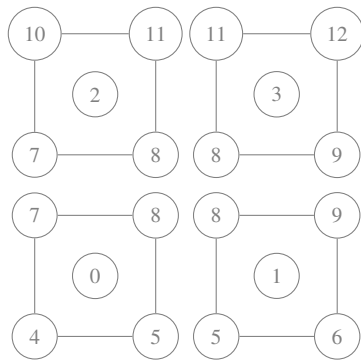
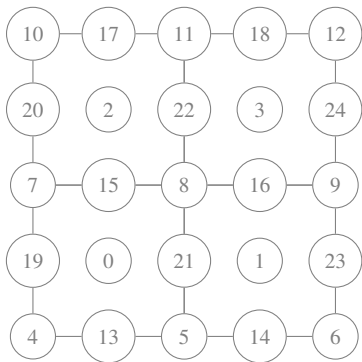
Mesh Transformations

Things We Can Do

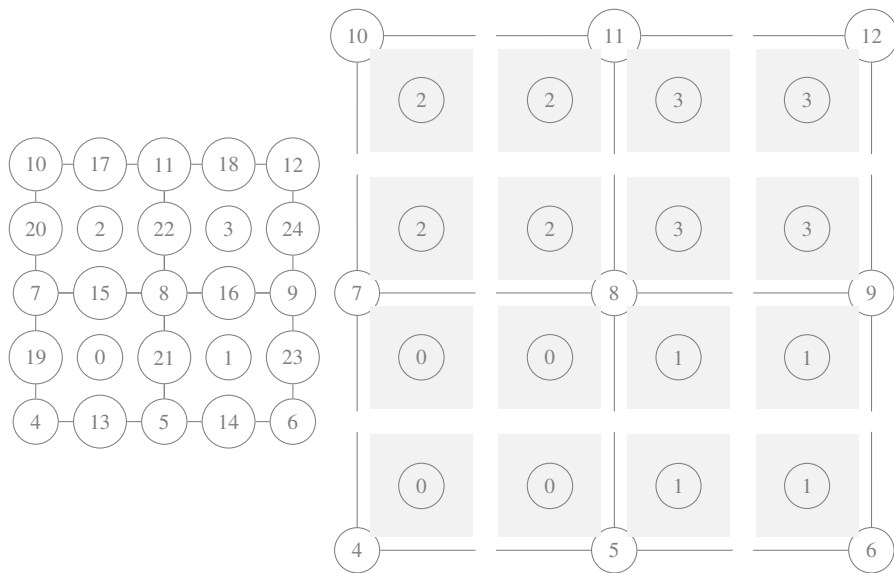
Things We Cannot Yet Do

There is an isomorphism
between meshes and DAGs.

Closure Decomposition



Star Decomposition



Hasse Diagram

Mesh Representation (Knepley and Karpeev 2009; Lange et al. 2016)
for a CW-complex (Whitehead 1949)

DAG vertices $\longleftrightarrow$ k -cells

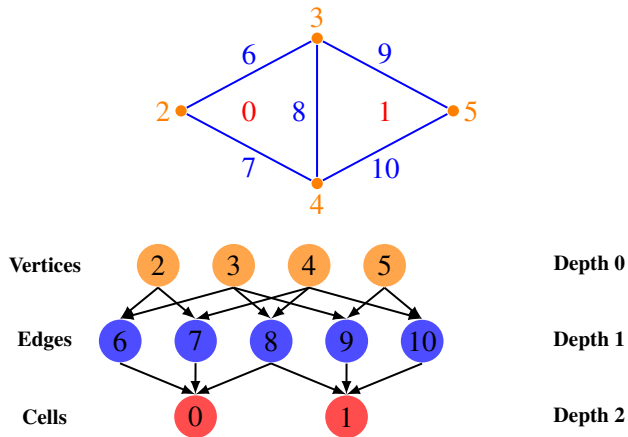
DAG edges $\longleftrightarrow$ boundary adjacency

Edge labels $\longleftrightarrow$ dihedral group representers

We call each DAG vertex a *mesh point*

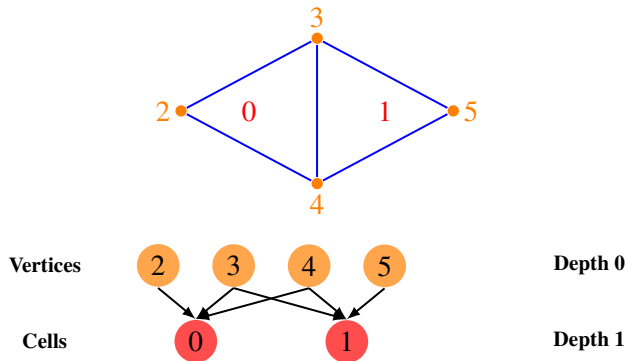
Sample Meshes

Interpolated triangular mesh



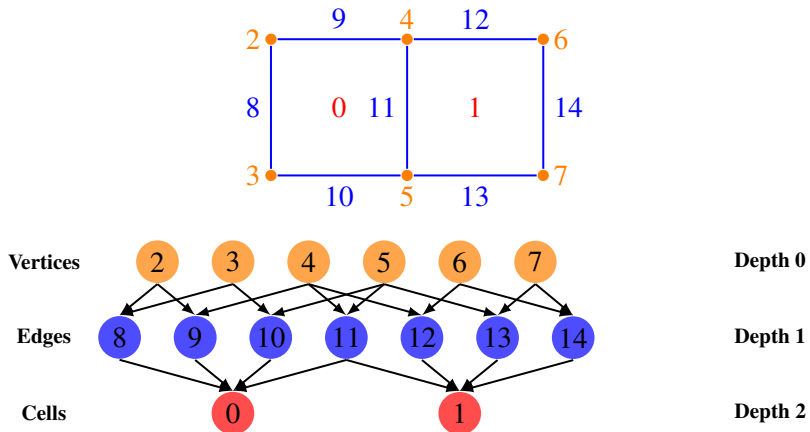
Sample Meshes

Optimized triangular mesh



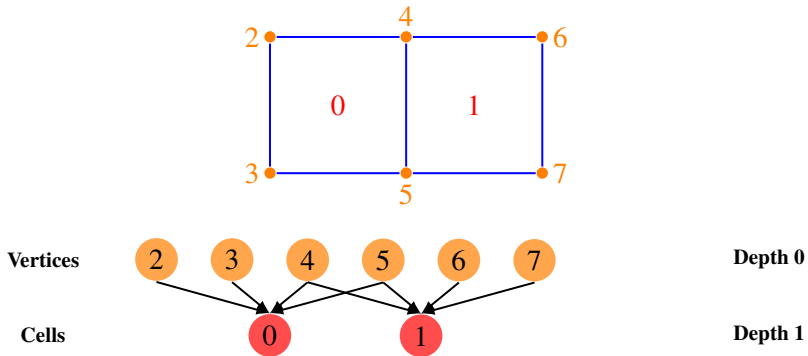
Sample Meshes

Interpolated quadrilateral mesh



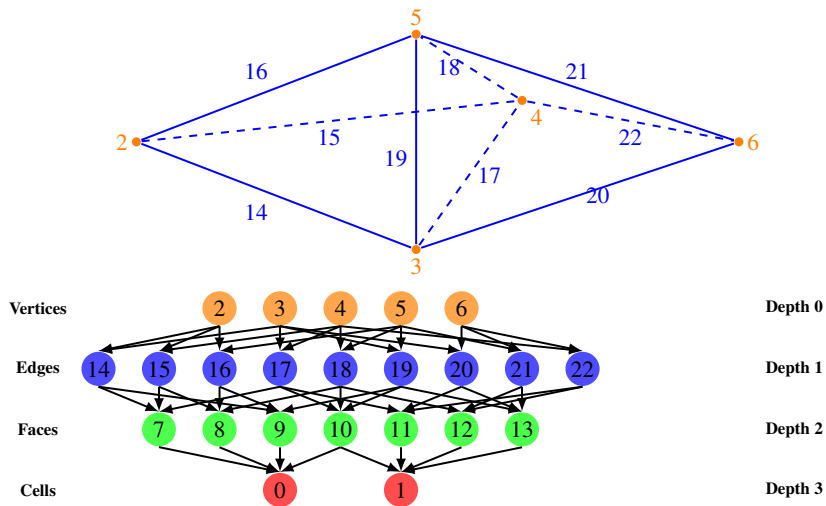
Sample Meshes

Optimized quadrilateral mesh



Sample Meshes

Interpolated tetrahedral mesh

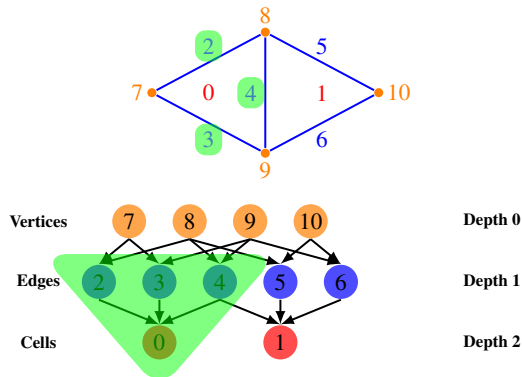


Basic Operations

Cone

We begin with the basic covering relation,

$$\text{cone}(0) = \{2, 3, 4\}$$

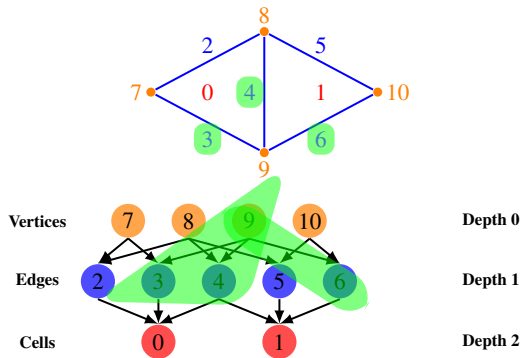


Basic Operations

Support

reverse arrows to get the
dual operation,

$\text{support}(9) = \{3, 4, 6\}$

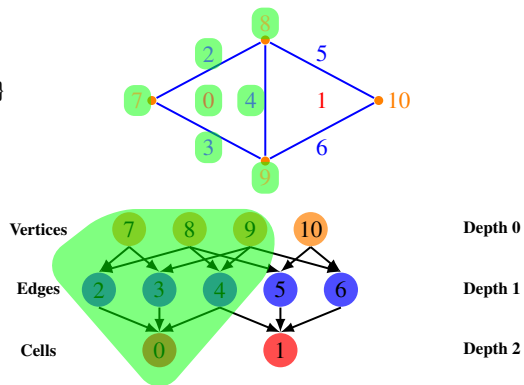


Basic Operations

Closure

add the transitive closure of
the relation,

$$\text{closure}(0) = \{0, 2, 3, 4, 7, 8, 9\}$$

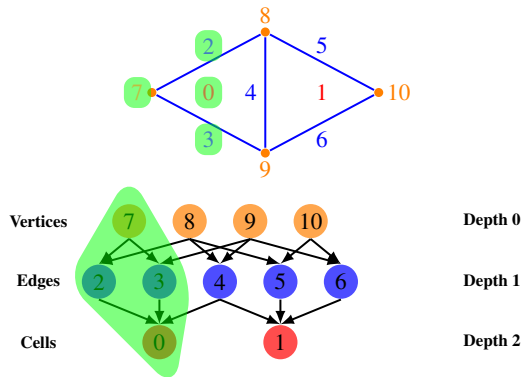


Basic Operations

Star

and the transitive closure of
the dual,

$$\text{star}(7) = \{7, 2, 3, 0\}$$

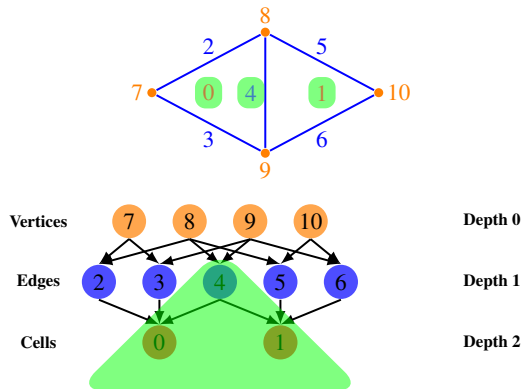


Basic Operations

Meet

and augment with lattice
operations.

$$\text{meet}(0, 1) = \{4\}$$

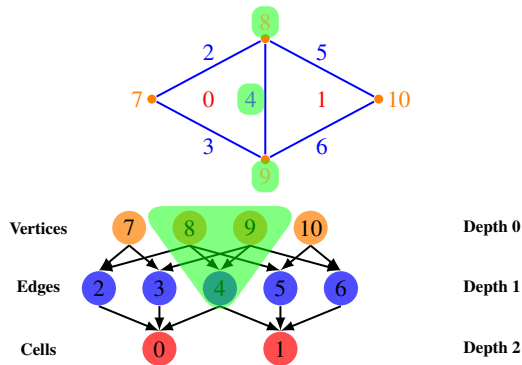


Basic Operations

Join

and augment with lattice operations.

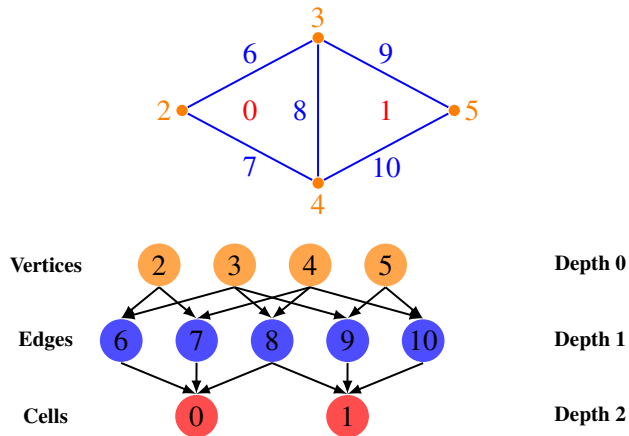
$$\text{join}(8, 9) = \{4\}$$



Sample Meshes

Interpolated triangular mesh

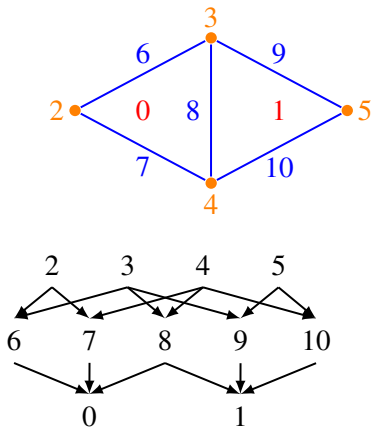
Starting with an interpolated mesh,



Sample Meshes

Interpolated triangular mesh

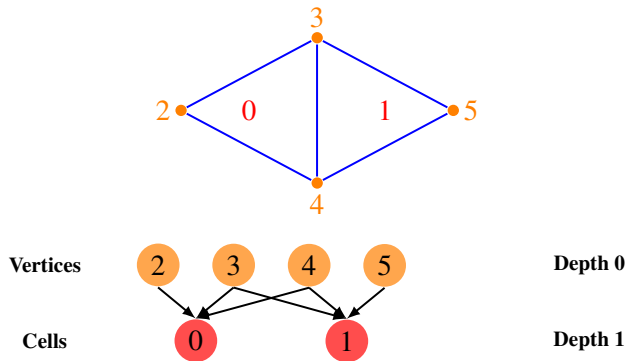
we can infer the depth in $\mathcal{O}(N)$ (BFS),



Sample Meshes

Interpolated triangular mesh

and we can infer the intermediate levels in $\mathcal{O}(N)$ (Join).



Cell Orientation

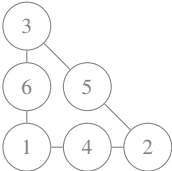
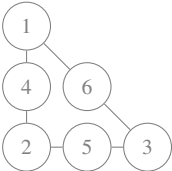
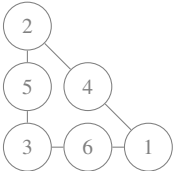
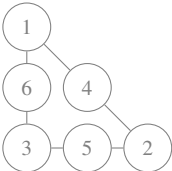
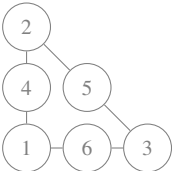
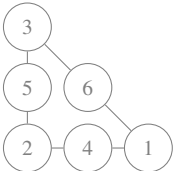
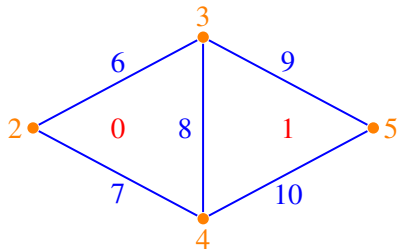
Orientation	0	1	2
Arrangement			
Orientation	-1	-2	-3
Arrangement			

Table: The dihedral group D_3 for the triangle

Orientation

Edge Decoration



Vertices



Depth 0

Edges



Depth 1

Cells



Depth 2

Outline

Mesh Representation

Mesh Transformations

Things We Can Do

Things We Cannot Yet Do

Transformation

Production Rule:

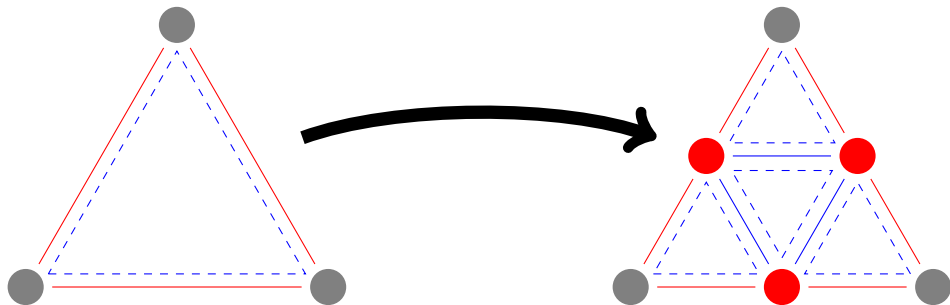
$$p \rightarrow \{(q_k, o_k)\}$$

or

$$q_k \in \text{child}(p)$$

Example Transformation

Refinement



Transformation Conditions

Condition 1:

$$\text{cone}(\text{child}(p)) \in \text{child}(\text{cl}(p)),$$

Condition 2:

$$\text{parent}(\text{st}(q)) \in \text{st}(\text{parent}(q))$$

Transformation Conditions

Condition 1:

$$\text{cone}(\text{child}(p)) \in \text{child}(\text{cl}(p)),$$

Condition 2:

$$|\text{parent}(q)| = 1$$

Closure Complexity

Consider a point q' in the cone of q , so that

$$\begin{aligned} q' &\in \text{cone}(q) \\ &\in \text{child}(\text{cl}(p)) \end{aligned}$$

by Condition 1, meaning

$$\exists p' \in \text{cl}(p), q' \in \text{child}(p').$$

We may take the cone of each side and use Condition 1 again,

$$\begin{aligned} \text{cone}(q') &\in \text{cone}(\text{child}(p')) \\ &\in \text{child}(\text{cl}(p')) \\ &\in \text{child}(\text{cl}(p)) \end{aligned}$$

Thus

$$\text{cl}(\text{child}(p)) \in \text{child}(\text{cl}(p)).$$

Support Complexity

Let p produce q , and consider a point q' in the star of q ,

$$q' \in \text{st}(q) \iff q \in \text{cl}(q')$$

Let q' be produced by a point p' ,

$$\begin{aligned} q' &\in \text{child}(p') \\ \text{cl}(q') &\in \text{child}(\text{cl}(p')) \\ q &\in \text{child}(\text{cl}(p')) \end{aligned}$$

Using Condition 2,

$$\begin{aligned} \text{parent}(q) &\in \text{cl}(p') \\ p &\in \text{cl}(p') \\ p' &\in \text{st}(p) \end{aligned}$$

and thus

$$\text{st}(\text{child}(p)) \in \text{child}(\text{st}(p)).$$

Numbering

With unique parents,
we can number everything *a priori*,
in parallel.

Output Sensitivity

To generate a closure in the output mesh,
we need only the producing point closure.

To generate a star in the output mesh,
we need only the producing point star.

Outline

Mesh Representation

Mesh Transformations

Things We Can Do

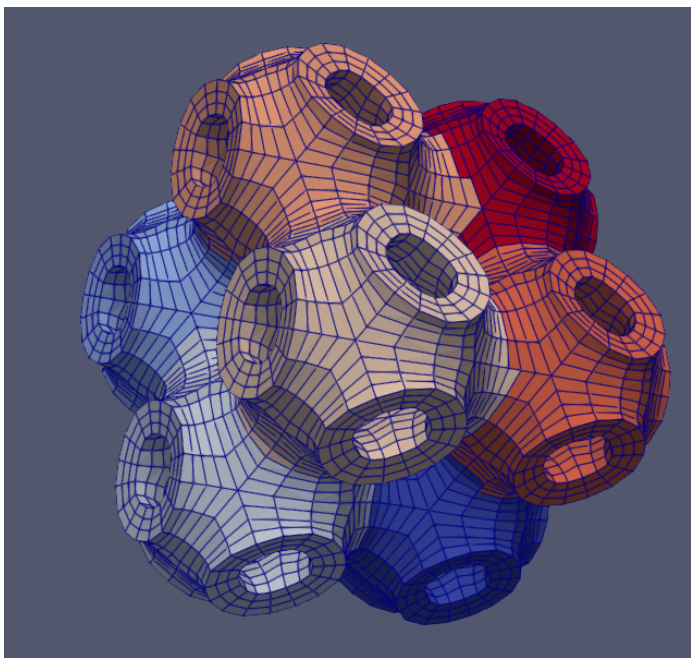
Things We Cannot Yet Do

Implementations

- ▶ Filtering

Implementations

- ▶ Filtering
- ▶ Extrusion
 - ▶ From marked surfaces

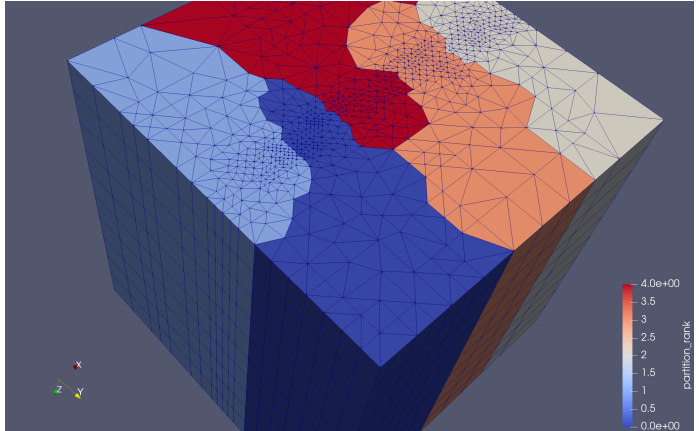


Implementations

- ▶ Filtering
- ▶ Extrusion
 - ▶ From marked surfaces
- ▶ Cell Conversion

Implementations

- ▶ Filtering
- ▶ Extrusion
 - ▶ From marked surfaces
- ▶ Cell Conversion
- ▶ Refinement ...



Refinement

- ▶ Regular refinement

Refinement

- ▶ Regular refinement
- ▶ Alfeld refinement

Refinement

- ▶ Regular refinement
- ▶ Alfeld refinement
- ▶ Boundary Layer refinement
 - ▶ On marked surfaces

Refinement

- ▶ Regular refinement
- ▶ Alfeld refinement
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- ▶ SBR refinement (Plaza-Carey)
 - ▶ Initialized at marked cells

Refinement

- ▶ Regular refinement
- ▶ Alfeld refinement
- ▶ Boundary Layer refinement
 - ▶ On marked surfaces
- ▶ SBR refinement (Plaza-Carey)
 - ▶ Initialized at marked cells
- ▶ Composition ...

LOD

1. Select mesh patch with filter

LOD

1. Select mesh patch with filter
2. Refine patch

LOD

1. Select mesh patch with filter
2. Refine patch
3. Solve on patch

LOD

1. Select mesh patch with filter
2. Refine patch
3. Solve on patch
4. Assemble into global corrector
 - ▶ Numbering is from an ephemeral global fine mesh

LOD

1. Select mesh patch with filter
2. Refine patch
3. Solve on patch
4. Project fine element matrix to new coarse space
5. Assemble new coarse A

Outline

Mesh Representation

Mesh Transformations

Things We Can Do

Things We Cannot Yet Do

Theory

- ▶ Calculate production tables using algebraic topology
- ▶ Anisotropic refinement
- ▶ Coarsening

References I



Knepley, Matthew G. and Dmitry A. Karpeev (2009). “Mesh Algorithms for PDE with Sieve I: Mesh Distribution”. In: [Scientific Programming](#) 17.3. <http://arxiv.org/abs/0908.4427>, pp. 215–230. DOI: 10.3233/SPR-2009-0249. URL: <http://arxiv.org/abs/0908.4427>.



Lange, Michael, Lawrence Mitchell, Matthew G. Knepley, and Gerard J. Gorman (2016). “Efficient mesh management in Firedrake using PETSc-DMPlex”. In: [SIAM Journal on Scientific Computing](#) 38.5, S143–S155. DOI: 10.1137/15M1026092. eprint: <http://arxiv.org/abs/1506.07749>.



Whitehead, John HC (1949). “Combinatorial homotopy. I”. In: [Bulletin of the American Mathematical Society](#) 55.3. P1, pp. 213–245.