Ephemeral Meshes

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Meshes should consist of

data *and* code.

Outline

Mesh Representation

Mesh Transformations

Things We Can Do

Things We Cannot Yet Do

There is an isomorphism

between meshes and DAGs.

Hasse Diagram

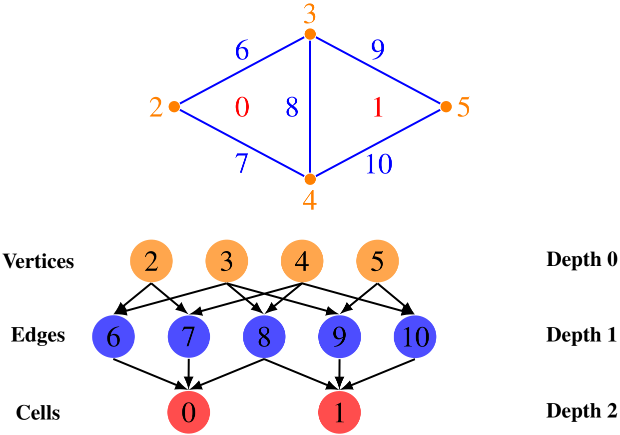
Mesh Representation (Knepley and Karpeev 2009; Lange et al. 2016)
for a CW-complex (Whitehead 1949)

DAG vertices	$\longleftrightarrow$	k-cells
DAG edges	$\longleftrightarrow$	boundary adjacency
Edge labels	$\longleftrightarrow$	dihedral group representers

We call each DAG vertex a *mesh point*

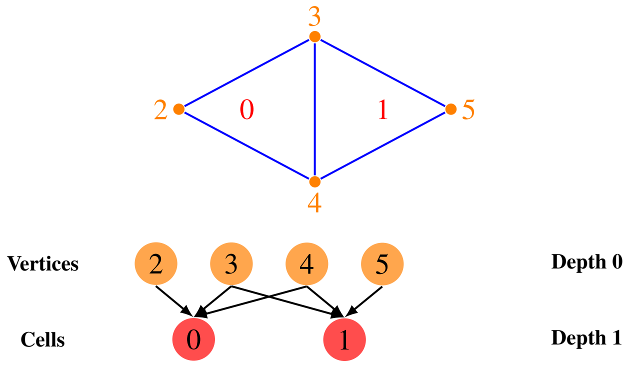
Sample Meshes

Interpolated triangular mesh



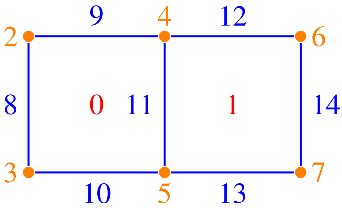
Sample Meshes

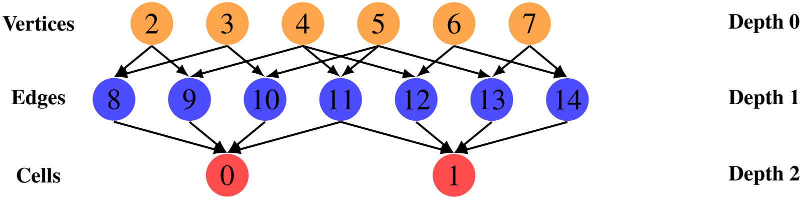
Optimized triangular mesh



Sample Meshes

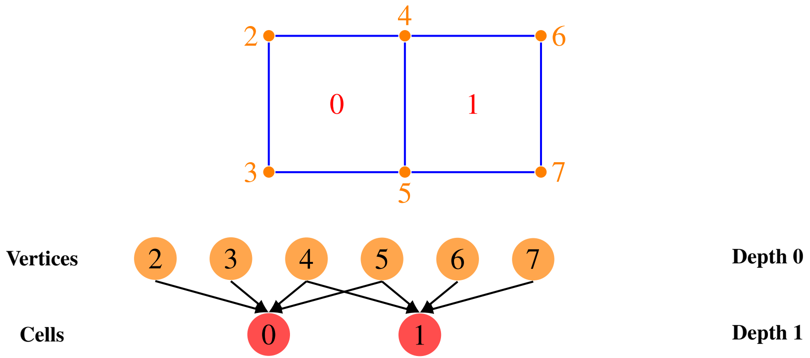
Interpolated quadrilateral mesh





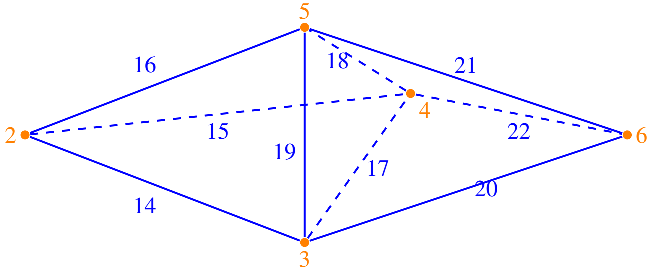
Sample Meshes

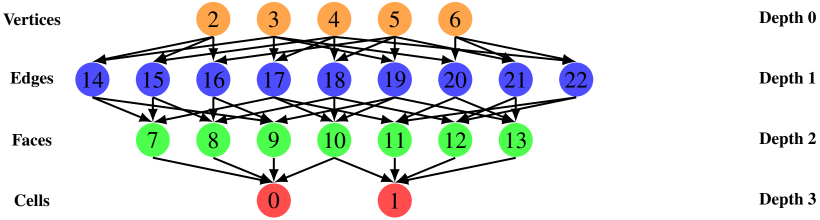
Optimized quadrilateral mesh



Sample Meshes

Interpolated tetrahedral mesh



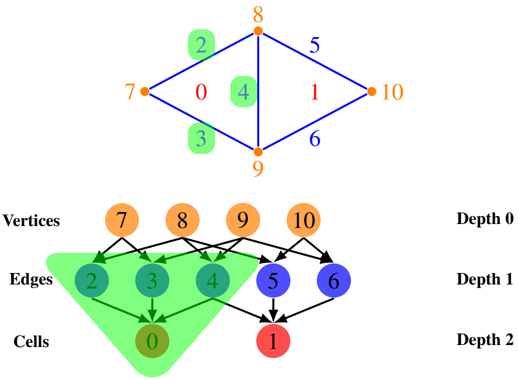


Basic Operations

Cone

We begin with the basic covering relation,

cone(0) = {2, 3, 4}

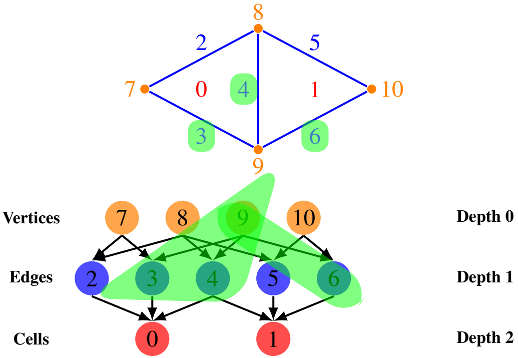


Basic Operations

Support

reverse arrows to get the dual operation,

support(9) = {3, 4, 6}

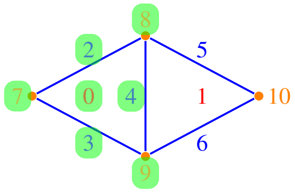


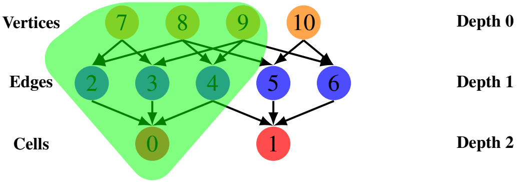
Basic Operations

Closure

add the transitive closure of the relation,

closure(0) = {0, 2, 3, 4, 7, 8, 9}

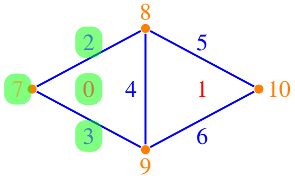




Basic Operations
Star

and the transitive closure of the dual,

star(7) = {7, 2, 3, 0}

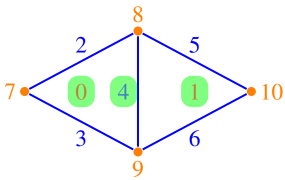


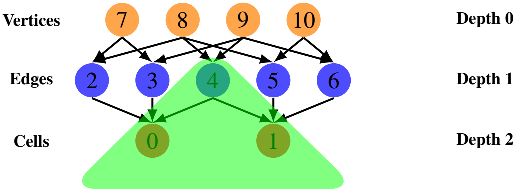
Basic Operations

Meet

and augment with lattice operations.

meet(0, 1) = {4}



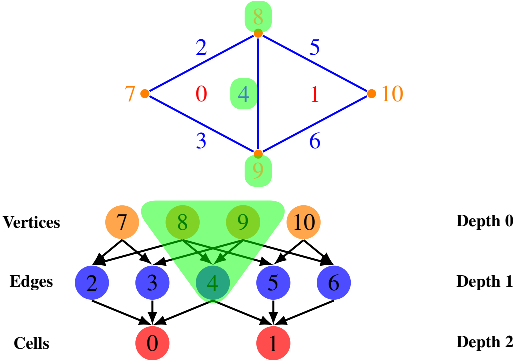


Basic Operations

Join

and augment with lattice operations.

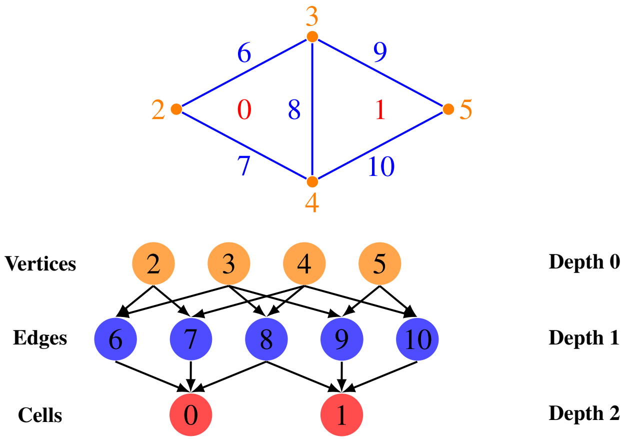
join(8, 9) = {4}



Sample Meshes

Interpolated triangular mesh

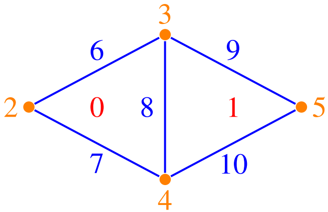
Starting with an interpolated mesh,

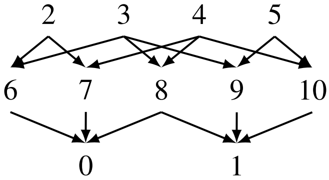


Sample Meshes

Interpolated triangular mesh

we can infer the depth in $\mathcal{O}(N)$ (BFS),

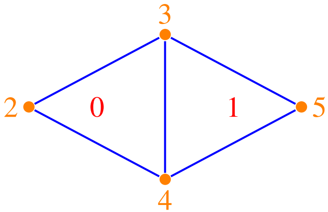


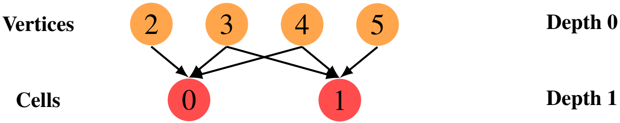


Sample Meshes

Interpolated triangular mesh

and we can infer the intermediate levels in $\mathcal{O}(N)$ (Join).





Cell Orientation

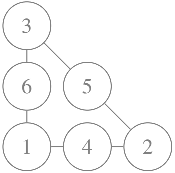
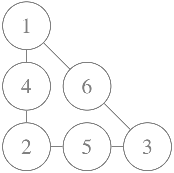
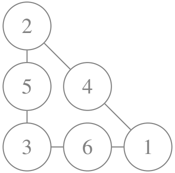
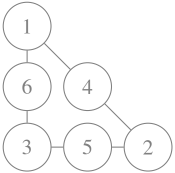
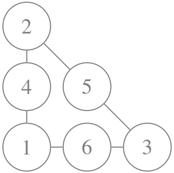
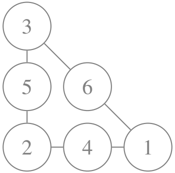
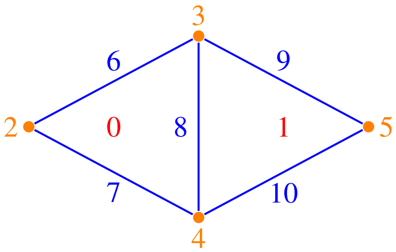
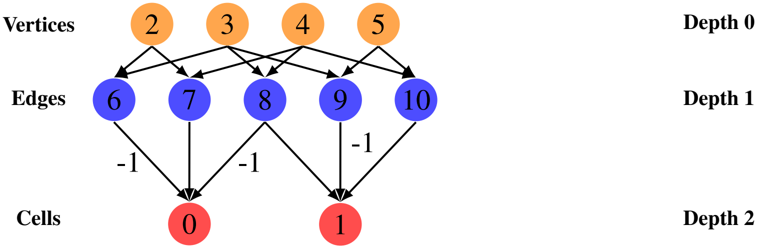
Orientation	0	1	2
Arrangement			
Orientation	**-1**	**-2**	**-3**
Arrangement			

Table: The dihedral group D_3 for the triangle

Orientation

Edge Decoration





Outline

Transformation

Production Rule:

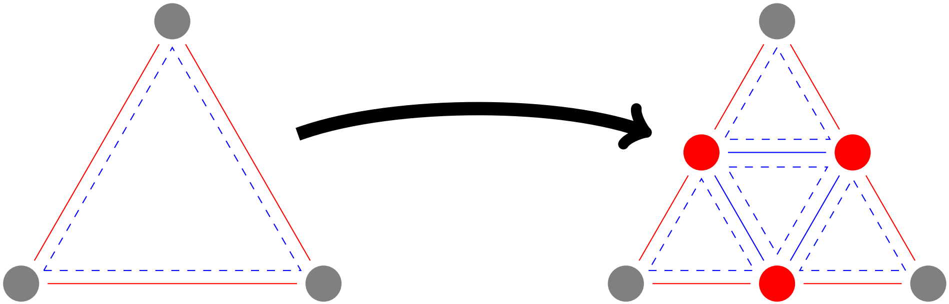
$$p \to \{(q_k, o_k)\}$$

or

$$q_k \in \mathbb{child}(p)$$

Example Transformation

Refinement



Transformation Conditions

Condition 1:

$$\mathbb{cone}(\mathbb{child}(p)) \in \mathbb{child}(\mathbb{cl}(p)),$$

Condition 2:

$$\mathbb{parent}(\mathbb{st}(q)) \in \mathbb{st}(\mathbb{parent}(q))$$

Transformation Conditions

Condition 1:

$$\mathbb{cone}(\mathbb{child}(p)) \in \mathbb{child}(\mathbb{cl}(p)),$$

Condition 2:

$$|\mathbb{parent}(q)| = 1$$

Closure Complexity

Consider a point q' in the cone of q, so that

$$\begin{aligned} q' &\in \mathbb{cone}(q) \\ &\in \mathbb{child}(\mathbb{cl}(p)) \end{aligned}$$

by Condition 1, meaning

$$\exists p' \in \mathbb{cl}(p), q' \in \mathbb{child}(p').$$

We may take the cone of each side and use Condition 1 again,

$$\begin{aligned} \mathbb{cone}(q') &\in \mathbb{cone}(\mathbb{child}(p')) \\ &\in \mathbb{child}(\mathbb{cl}(p')) \\ &\in \mathbb{child}(\mathbb{cl}(p)) \end{aligned}$$

Thus

$$\mathbb{cl}(\mathbb{child}(p)) \in \mathbb{child}(\mathbb{cl}(p)).$$

Support Complexity

Let p produce q, and consider a point q' in the star of q,

$$q' \in \mathbb{st}(q) \iff q \in \mathbb{cl}(q')$$

Let q' be produced by a point p',

$$\begin{aligned} q' &\in \mathbb{child}(p') \\ \mathbb{cl}(q') &\in \mathbb{child}(\mathbb{cl}(p')) \\ q &\in \mathbb{child}(\mathbb{cl}(p')) \end{aligned}$$

Using Condition 2,

$$\begin{aligned} \mathbb{parent}(q) &\in \mathbb{cl}(p') \\ p &\in \mathbb{cl}(p') \\ p' &\in \mathbb{st}(p) \end{aligned}$$

and thus

$$\mathbb{st}(\mathbb{child}(p)) \in \mathbb{child}(\mathbb{st}(p)).$$

Numbering

With unique parents,

we can number everything *a priori*,

in parallel.

Output Sensitivity

To generate a closure in the output mesh,
we need only the producing point closure.

To generate a star in the output mesh,
we need only the producing point star.

Outline

Implementations

- Filtering

Implementations

- Filering
- Extrusion
  - From marked surfaces

Implementations

- Filering
- Extrusion
  - From marked surfaces
- Cell Conversion

Implementations

- Filtering
- Extrusion
  - From marked surfaces
- Cell Conversion
- Refinement ...

Refinement

- Regular refinement

Refinement

- Regular refinement
- Alfeld refinement

Refinement

- Regular refinement
- Alfeld refinement
- Boundary Layer refinement
  - On marked surfaces

Refinement

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- Alfeld refinement
- Boundary Layer refinement
  - On marked surfaces
- SBR refinement (Plaza-Carey)
  - Initialized at marked cells

Refinement

- Regular refinement
- Alfeld refinement
- Boundary Layer refinement
  - On marked surfaces
- SBR refinement (Plaza-Carey)
  - Initialized at marked cells
- Composition …

LOD

1. Select mesh patch with filter

LOD

1. Select mesh patch with filter
2. Refine patch

LOD

1. Select mesh patch with filter
2. Refine patch
3. Solve on patch

LOD

1. Select mesh patch with filter
2. Refine patch
3. Solve on patch
4. Assemble into global corrector
   - Numbering is from an ephemeral global fine mesh

LOD

1. Select mesh patch with filter
2. Refine patch
3. Solve on patch
4. Project fine element matrix to new coarse space
5. Assemble new coarse A

Outline

Mesh Representation

Mesh Transformations

Things We Can Do

Things We Cannot Yet Do

Theory

- Calculate production tables using algebraic topology
- Anisotropic refinement
- Coarsening

References I

Knepley, Matthew G. and Dmitry A. Karpeev (2009). “Mesh Algorithms for PDE with Sieve I: Mesh Distribution”. In: Scientific Programming 17.3. http://arxiv.org/abs/0908.4427, pp. 215–230. DOI: 10.3233/SPR-2009-0249. URL: http://arxiv.org/abs/0908.4427.

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Whitehead, John HC (1949). “Combinatorial homotopy. I”. In: Bulletin of the American Mathematical Society 55.3. P1, pp. 213–245.