APR 19

Weighted Interval Scheduling problem

<u>Input:</u> n intervals; ith interval $= (s_i, f_i, v_i)$

start time, finish time, value.

<u>Output:</u> Instead of outputting an optimal schedule O, output

$$V(O) = \sum_{i \in O} v_i$$

<u>Def.</u> $OPT(j)$ = value of an optimal solution for $[j] \to \begin{cases} (s_1, f_2, v_1) \\ \vdots \\ (s_j, f_j, v_j) \end{cases}$

$1 \leq j \leq n$

$f(1) \leq f(2) \leq \cdots \leq f(n)$

<u>Goal:</u> $OPT(n)$

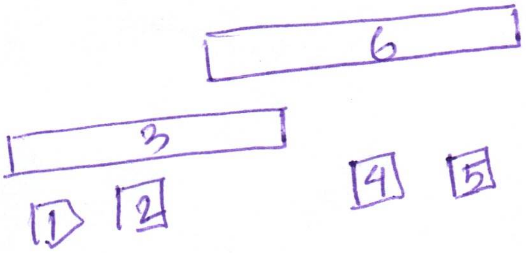
<u>Def:</u> let O_j be an optimal solution for $[j]$

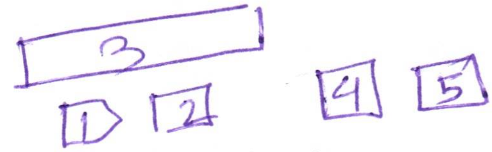
$$V(O_j) = OPT(j) \quad (*)$$

<u>Case 1:</u> $j \notin O_j$

$6 \notin O_6$

O_6 is an optimal solution for $[5]$





$OPT(6) = OPT(6-1)$

$OPT(j) = OPT(j-1)$

Case 2: $j \in O_j$ | $6 \in O_6$

$\boxed{1}$ $\boxed{2}$

$O_6 \setminus \{6\}$ is an optimal solution for $[2]$.

O_6: 6, 1 2 (OPT(6))

$\cup|$

$O_6 \setminus \{6\}$: 1 2 (OPT(2))

$$OPT(6) = V_6 + OPT(2)$$
$$OPT(6) = V_6 + OPT(P(6)) \quad \rightarrow P(6) = 2$$

$P(j)$: largest value of $i < j$ that doesn't conflict with j.

$$\rightarrow OPT(j) = V_j + OPT(P(j))$$

$$OPT(j) = \max\{\underbrace{OPT(j-1)}_{j \notin O_j}, \underbrace{V_j + OPT(P(j))}_{j \in O_j}\}$$

Case 1: $j \notin O_j$

Claim: O_j is an optimal solution for $[j-1]$

$$OPT(j) \underset{\text{by } (*)}{=} V(O_j) \underset{\text{claim}}{=} OPT(j-1)$$

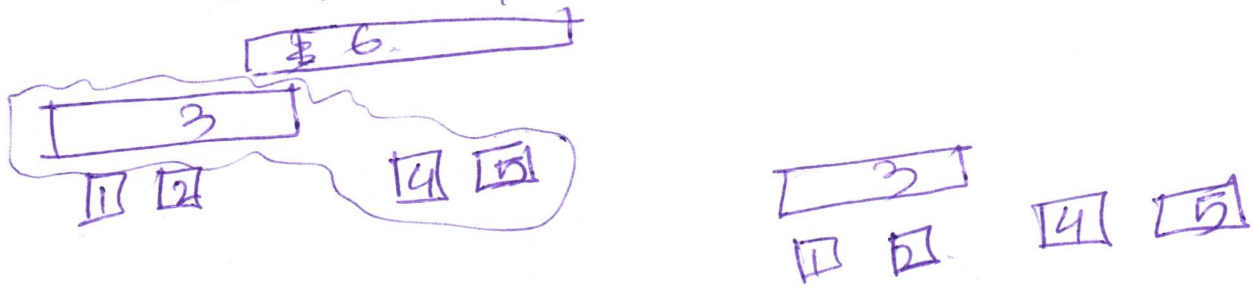
Pf. (Idea): By contradiction:

Assume O_j is NOT an optimal solution for $[j-1]$.

$\Rightarrow \exists$ a feasible/valid schedule O' for $[j-1]$

AND $V(O') > V(O_j)$.

Q: is O' feasible/Valid for $[j]$?



A: O' is also ~~opt~~ feasible for $[j]$.

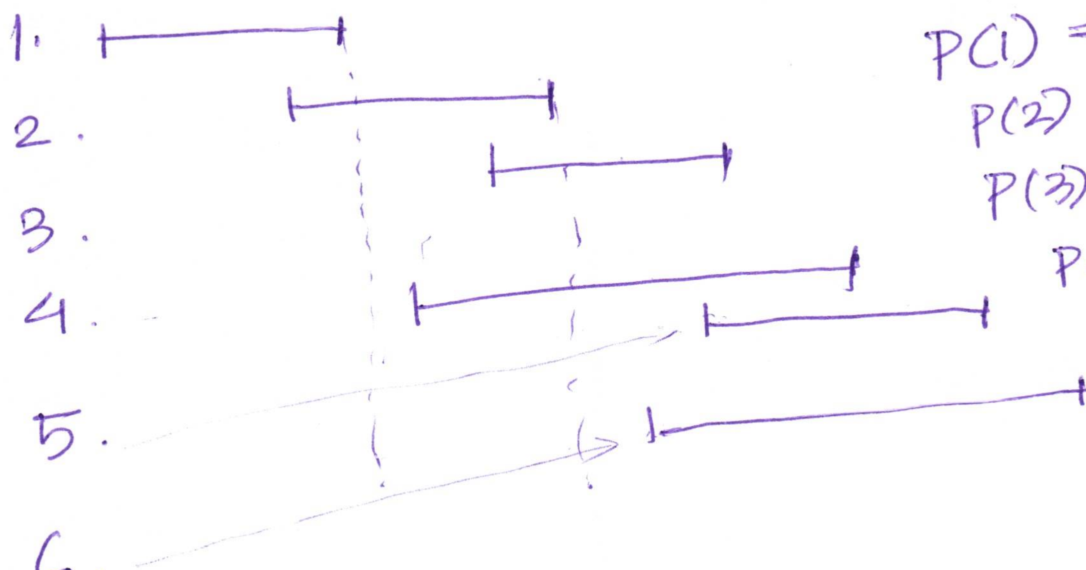
$\Rightarrow$ Contradicts the optimality of O_j.

$\Rightarrow$ as now we have another feasible solution O' AND $v(O') > v(O_j)$. $\square$

Case 2: $j \in O_j$.

$$P(j) = \begin{cases} \text{largest index } i < j \text{ s.t. } i \,\&\, j \text{ don't conflict.} \\ 0, \quad o/w \end{cases}$$

n=6



$P(1) = 0$
$P(2) = 0$
$P(3) = 1$
$P(4) = 1$
$P(5) = 2$
$P(6) = 2$

Note: (1) $P(j)+1, \ldots, j-1$ conflict with j.
(2) $1, \ldots, P(j)$ do not " " j

$\Rightarrow$ If we pick $j \in O_j$, the remaining subproblem $1, \ldots, P(j)$

Claim: $O_j \setminus \{j\}$ is an optimal solution for $[P(j)]$

Pf. (idea): By contradiction.

$\Rightarrow \exists$ a feasible solution O' for $[P(j)]$

AND $v(O') > v(O_j \setminus \{j\})$.

Note: $O' \cup \{j\}$ is valid for $[j]$.

$$
\begin{aligned}
v(O' \cup \{j\}) &= v(O') + v_j \\
&> v(O_j \setminus \{j\}) + v_j \\
&= v(O_j) - v_j + v_j \\
&= v(O_j)
\end{aligned}
$$

$\Rightarrow$ contradicts the optimality of O_j. $\square$