Rule of thumb: for asymptotics of $T(n)$ enough to show: $T(\lfloor x \rfloor) \to T(x)$, $T(\lceil x \rceil) \to T(x)$

$$\Rightarrow T(n) \leq \begin{cases} c, & \text{if } n=1 \\ cn + 2T(n/2), & \text{o/w} \end{cases}$$

Lemma: $T(n) \leq cn\log_2 n + cn$ $\left(\leq O(n\log n)\right)$

$\Rightarrow$ Merge Sort runs in $O(n\log n)$ time.

Remarks: Apr 5

1. $O(n\log n)$ is the best known upper bound for general sorting algo.
2. can do faster ($O(n)$ time) if the domain of a_i's is of size $O(n)$. (T/F #1)
3. Can do faster for "almost" sorted inputs.
4. Any comparison-based sorting algo ~~me~~ needs to make $\Omega(n\log n)$ comparisons.

Strategies for solving recurrence:

1. "Unroll" the recurrence & identify the pattern; use the pattern.

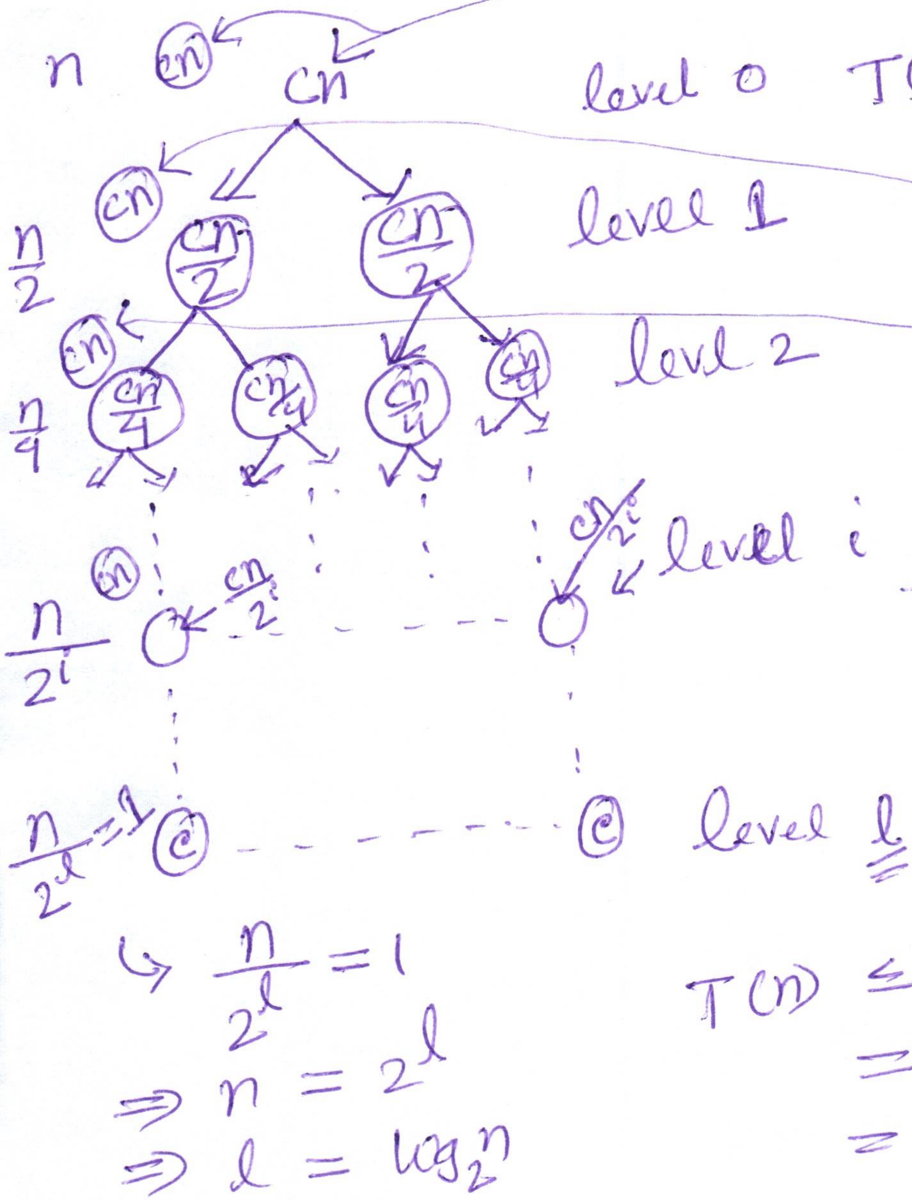
② Given the answer & verify by induction on n

$$T(n) \le \begin{cases} O(1), & \text{if } n = 1 \\ cn + 2T(n/2), & \text{o/w} \end{cases}$$

<u>Goal:</u> $T(n) \le cn \log_2 n + cn$

<u>Assume:</u> n is a power of 2.

<u>Strategy 1:</u> "unroll" + identify pattern + use pattern



$$\begin{aligned} T(n) &\le cn + 2T(n/2) \\ &\le cn + 2\left(\frac{cn}{2} + 2T(n/4)\right) \\ &= cn + cn + 4T(n/4) \\ &\le cn + cn + 4\left(\frac{cn}{4} + 2T(n/8)\right) \\ &= cn + cn + cn + 8T(n/8) \end{aligned}$$

→ Contribution from level i:

$$2^i \cdot \frac{cn}{2^i} = cn$$

$$\hookrightarrow \frac{n}{2^l} = 1$$

$$\Rightarrow n = 2^l$$

$$\Rightarrow l = \log_2 n$$

levels

$$\begin{aligned} T(n) &\le cn \cdot (\#\,\text{levels}) \\ &= cn(l+1) \\ &= cn(\log_2 n + 1) \\ &= cn \log_2 n + cn \end{aligned}$$

□

Strategy 2: Guess $T(n) \le cn\log_2 n + cn$ — (*)

& verify by induction on n.

Base case: $n=1$, need to show $c \le c \cdot 1 \log_2 1 + c \cdot 1$
$= c$

$\Rightarrow c \le c$ ✓

IH: (*) holds $n \leftarrow n/2$

$$\Rightarrow T(n/2) \le \frac{cn}{2}\log_2 \frac{n}{2} + \frac{cn}{2} = \frac{cn}{2}\left(\log_2 \frac{n}{2} + 1\right)$$
$$= \frac{cn}{2}\left(\log_2 n - \log_2 2 + 1\right)$$
$$= \frac{cn}{2}\left(\log_2 n - 1 + 1\right)$$
$$= \frac{cn}{2}\log_2 n$$

I.S.: By recurrence,

$$T(n) \le cn + 2T(n/2)$$
By I.H. $\rightarrow$ $$\le cn + 2\left(\frac{cn}{2}\log_2 n\right)$$
$$= cn + cn\log_2 n \quad \square$$