Mar 27

Lemma 1: If at the end of each iteration $u \in R$, the path P_u is a shortest s-u path.

s-u path in our Dijkstra tree

Pf (idea): By induction on $|R|$.

Base Case: $|R| = 1 \Rightarrow R = \{s\}$, $d(s) = 0$

IH: Assume lemma 1 is true for $|R| = k$, $k \geq 1$.

IS: Assume $|R| = k+1$.

Let ω is $(k+1)^{th}$ node added to R.

WLOG, assume u has discovered ω.

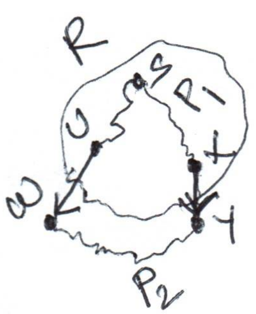
$P_\omega = P_{u,\omega}$ ← in our construction of Dijkstra tree.

$d(\omega) = d(u) + \ell(u, \omega)$

Goal: P_ω is a shortest s-ω path.

Pf (idea): By contradiction.

Assume $\exists$ an s-ω path $P'_\omega \neq P_\omega$ but $\ell(P'_\omega) < \ell(P_\omega)$. — (*)



As $s \in R$ but $w \notin R$, P'_w has to "cross" the boundary of R at some edge (x, y)

$$P'_w = P_1, x, y, P_2$$

$$\begin{aligned}
l(P'_w) &= l(P_1) + l(x,y) + l(P_2) \\
&\geq \underbrace{d(x) + l(x,y)}_{} + l(P_2) \\
&\geq d'(y) + l(P_2) \\
&\geq d'(y) \quad \text{as } l(P_2) \geq 0 \\
&\geq d'(w) \\
&= d(w) \\
&= l(P_w)
\end{aligned}$$

$d(x) \rightarrow d(x)$ is the ~~short~~ distance of x from s.

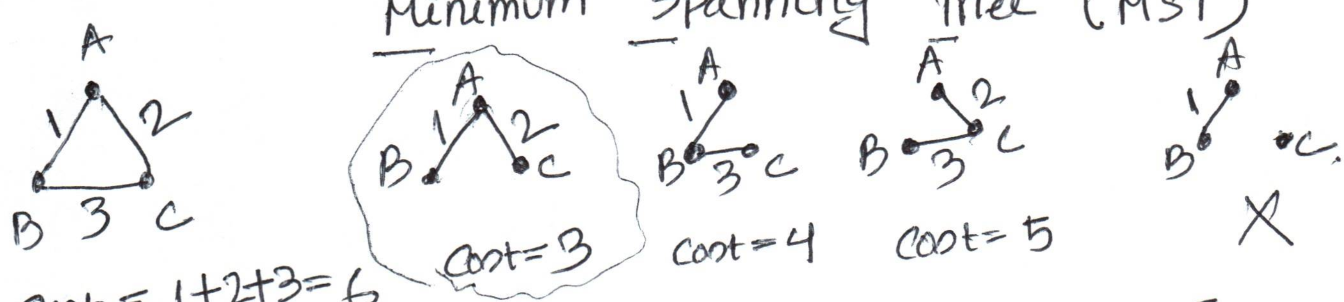
defn of $d'(y)$

Dijkstra's has chosen w over y

$\leftarrow$ algo. defn

$$\Rightarrow l(P'_w) \geq l(P_w) \Rightarrow \text{contradicts } (*) \quad \square$$

Minimum Spanning Tree (MST)



Input: $G = (V, E)$, $c_e \geq 0 \quad \forall e \in E$

undirected, connected

cost

for convenience only

OUTPUT: $E' \subseteq E$ s.t. T is a subgraph.

(i) $T = (V, E')$ is connected

(ii) $c(T) = \sum_{e \in T} c_e$ is minimized.

PROP: let $c_e > 0$, then any optimal solution $T = (V, E')$ is a tree.