Nov 10

Redux: Given a SAT formula $\varphi = c_1, \ldots, c_m$

$$\varphi \xrightarrow[\text{poly-time}]{} (G; m) \text{ s.t.}$$

φ is satisfiable $\iff$ G has an IS of size $\geq m$

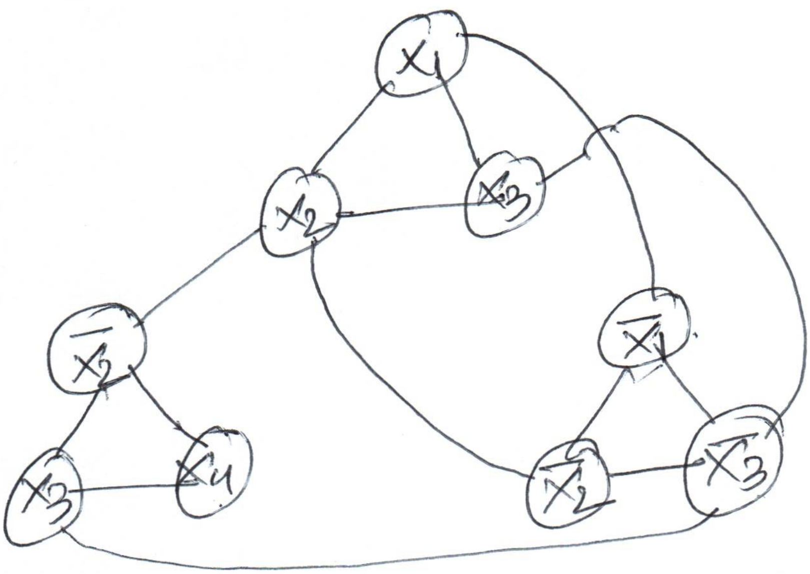
$\varphi: c_1 \wedge c_2 \wedge c_3$

$C_1 = x_1 \vee x_2 \vee x_3$ ✓

$C_2 = \overline{x_2} \vee x_3 \vee x_4$ ✓

$C_3 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$ ✓

Step 1: Convert each clause to their corresponding gadget.



pick one node from each gadget → All IS of size 3.

IS: $\{x_1, x_4, \overline{x_2}\}$

$\Rightarrow (1, 0, ?, 1)$
$x_1 \quad x_2 \quad x_3 \quad x_4$

Step 2: Add edges between nodes x_i and $\overline{x_i}$ for some i.

$\Rightarrow$ φ is satisfiable $\iff$ G (as constructed above) has an IS of size $\geq m$.

$m=3$

Def: X is NP complete if

1. $X \in NP$
2. $\forall Y \in NP,\ Y \le_p X$

THM 1: 3-SAT is NP complete. (Book)

lemma 1: X is NP-complete.
$\Rightarrow$ $X \in P \iff P = NP$

Pf. (idea):

$\Rightarrow$ Assume $X \in P$
X is NP-complete.
$\Rightarrow \forall Y \in NP,\ Y \le_p X$
$\Rightarrow$ as $X \in P$, $\forall Y \in NP$, $Y \in P$
$\Rightarrow NP = P$

$\Leftarrow$ Assume $P = NP$
X is NP-complete.
$\Rightarrow X \in NP$
$\Rightarrow$ as $NP = P$, $X \in P$. $\square$

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lemma 2: Y is NP-complete. $X \in NP$.
$Y \le_p X \Rightarrow X$ is NPC.

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cor 1: IS is NP complete.
Follows from $\rightarrow$ (IS $\in NP$ + THM 1 + 3SAT $\le_p$ IS + lemma 2)

~~cla~~ cor1: VC is NP complete. → (VC ∈ NP + cor 1 + IS $\leq_p$ VC + lemma 2)

General strategy to prove
X is NP-complete:

1. Show X ∈ NP (E.g., X = IS)
2. Identify an NP complete problem Y (e.g. Y = 3-SAT)
3. Show $Y \leq_p X$ (e.g., 3-SAT $\leq_p$ IS)

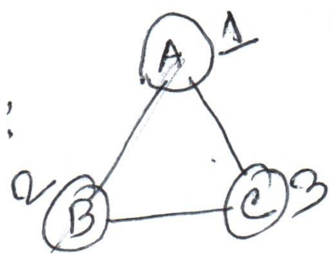
K-colorability / K-coloring

1: R
2: B
3: G

$G = (V, E)$

<u>Def</u>: Given a graph $G = (V, E)$, a K-coloring of this graph is a mapping $c: V \to \{1, \ldots, K\}$ s.t. $\forall (u, w) \in E$, $c(u) \neq c(w)$.

G_0:



∃ a 3-coloring
No 2-coloring for G_0.

Def (K-coloring problem):

<u>Input</u>: $G = (V, E)$; K

<u>Output</u>: T/~~F~~ 1 if G is K-colorable ← ∃ a K-coloring
0, o/w

<u>Ex</u>: G_0; 3 ✓ G_0; 2 ✗

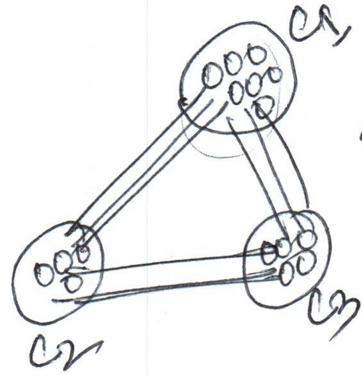
claim : K-colorability $\in$ NP.

↳ has an efficient verifier

witness : $C : V \to \{1, \ldots, K\}$

e.g. Co: $A \to 1$, $B \to 2$, $C \to 3$

e.g. 3-colorability.



← 3 partite

K-colorability $\equiv$ K-partite.

2-colorability $\in$ P → Uses the concept of bipartiteness → section 3 in KT.

THM: 3-SAT $\le_P$ 3-colorability $\le_P$ K-colorability (Book)

3-colorability is NP-complete.

claim + THM $\Rightarrow$ 3-colorability