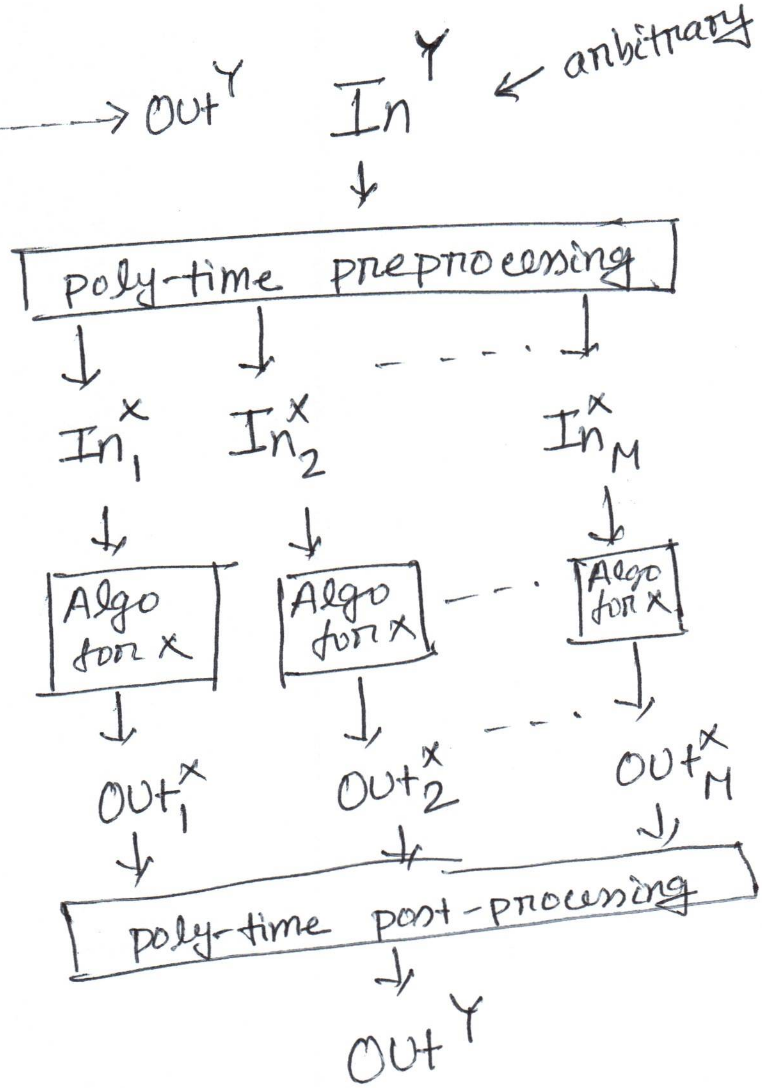
May 3

$$Y \le_p X$$

$\Rightarrow$ Y is poly-time reducible to X

$\Rightarrow$ $\exists$ a poly-time redux ~~to~~ from Y to X.

Solve: $In^Y \dashrightarrow Out^Y$



M=1 so far

$M = poly(N)$ — input size for Y

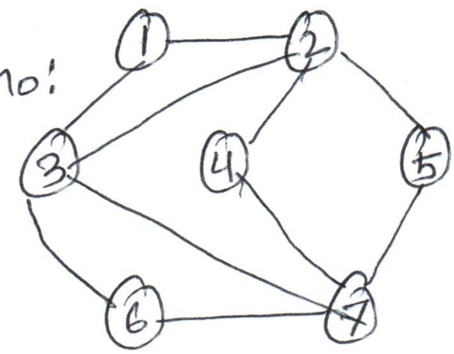
Ex: HW2 Q2 $\le_p$ Stable Matching

Going forward: consider only problems with binary outputs.

problem 1 Independent Set (IS) problem

Def: Given a graph $G=(V,E)$, ~~a~~ an IS is a subset $S \subseteq V$ s.t. NO edge exists between any pair of vertices in S.

G_0:



$\{1,4\}$ ✓ $\{2,5\}$ ✗ $\{1,4,7\}$ ✓ $\{3,4,5\}$ ✓, $\{1,4,5,6\}$ ✓

Input: $G=(V,E)$, $0 \le K \le n$ $(|V|=n)$

Output: TRUE if $\exists$ an IS of size $\ge K$.
FALSE otherwise.

~~G~~ Ex: G_0; 2 ✓T G_0; 3 ✓T G_0; 4 ✓T G_0; 5 ✗

[Note: Any subset of an IS is also an IS]
Ex:

Problem 2: Vertex Cover ~~(c.v.)~~ (V.C.)

Def: Given a graph $G=(V,E)$, a subset $C \subseteq V$ is a VC if every edge $e \in E$ has at least one endpoint in C. $\{1,2,3,4,5,6,7\}$ ✓ $\{2,3,7\}$ ✓ $\{1,7\}$ ✗ $\{1,2,3,4,5,6\}$, $\{1,2,6,7\}$ ✓

Input: $G=(V,E)$, $0 \le K \le n$

Output: TRUE if $\exists$ a VC of size $\le K$.

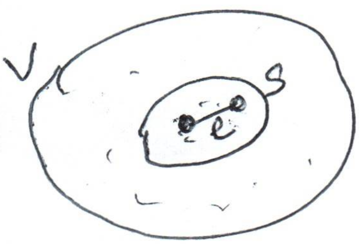
Ex: G_0; 6 ✓T G_0; 3 ✓T G_0; 2 ✗F

[Note: Any subset of size $n-1$ is a VC]
Ex:

THM (I) IS $\leq_p$ VC
(II) VC $\leq_p$ IS

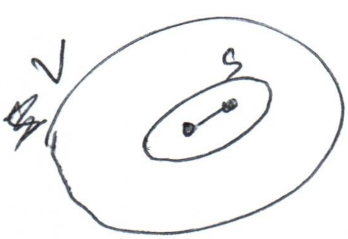
Lemma: $S \subseteq V$ is an IS $\iff$ $V \setminus S$ is a VC.

$\Rightarrow$ By contradiction.
For contradiction, assume that S is an IS, but $V \setminus S$ is not a VC.



$\Rightarrow$ $\exists$ an edge e that does NOT have any endpoints in $V \setminus S$.
$\Rightarrow$ e is completely inside S
$\Rightarrow$ both endpoints of e are in S
$\Rightarrow$ Contradicts S is an IS. $\square$

$\Leftarrow$ By contradiction.
For contradiction, assume that $V \setminus S$ is a VC, but S is not an IS.



$\Rightarrow$ $\exists$ an edge e between a pair of vertices in S.
$\Rightarrow$ e has both of its endpoints in S.
$\Rightarrow$ $V \setminus S$ does not have any endpoint of edge e.
$\Rightarrow$ contradicts $V \setminus S$ is a VC. $\square$

Cor: G has an IS of size $\geq k$
$\iff$ G has a VC of size $\leq n-k$.

THM 1 (I) IS $\le_p$ VC

Pf: Do a redux in poly-time.

Given ~~a~~ $(G; K)$ for an IS.

$$(G;K) \xrightarrow{(G;\, n-K)} \boxed{\text{Algo for VC}} \xrightarrow{T/F} \text{same } T/F$$

(II) VC $\le_p$ IS $\rightarrow$ similar proof.

Satisfiability / SAT problem

"prototypical" problem.

CNF formulas

SAT formula:

Conjunction / AND of clauses.

$\hookrightarrow$ Disjunction / OR of literals.

$\hookrightarrow x_i, \overline{x_i}$

$x_i \in \{0,1\}$

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$$

OR AND

$\hookrightarrow$ evaluate to T?

Q: $\exists$ an assignment ($x_i \in \{0,1\}$) s.t. SAT formula evaluates to T/1?