Lecture 8

CSE 331

Gale-Shapley Algorithm

Intially all men and women are free

While there exists a free woman who can propose

Let w be such a woman and m be the best man she has not proposed to

w proposes to m

If m is free

(m,w) get engaged

Else (m,w') are engaged

If m prefers w' to w

w remains free

Else

(m,w) get engaged and w' is free

Output the engaged pairs as the final output

The Lemmas

Lemma 1: The GS algorithm has at most n^2 iterations

Lemma 2: S is a perfect matching

Lemma 3: S has no instability

Proof Details of Lemma 1

Gale Shapley algorithm terminates

This page collects material from Fall 17 incarnation of CSE 331, where we proof details for the claim that the Gale-Shapley algorithm terminates in $O(n^2)$ iterations.

Where does the textbook talk about this?

`Section 1.1` in the textbook has the argument (though not in as much detail as below).

Fall 2017 material

Here is the lecture video (it starts from the part where we d the proof details):

Proof by contradiction

Assume the negation of what you want to prove

After some reasoning

Source: 4simpsons.wordpress.com

Two obervations

Obs 1: Once m is engaged he keeps getting engaged to "better" women

Obs 2: If w proposes to m' first and then to m (or never proposes to m) then she prefers m' to m

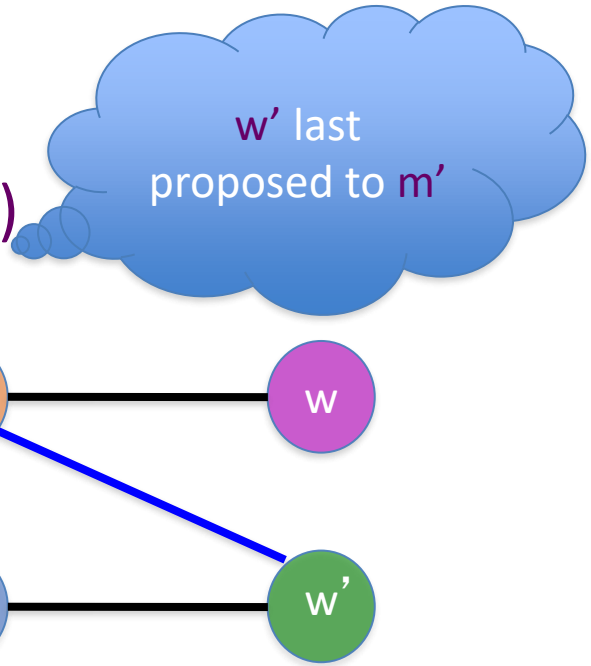
Proof of Lemma 3

By contradiction

Assume there is an instability (m,w’)

m prefers w’ to w

w’ prefers m to m’



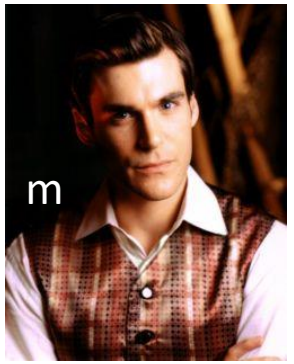
Contradiction by Case Analysis

Depending on whether w’ had proposed to m or not

Case 1: w’ never proposed to m

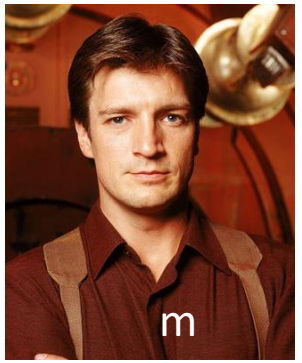
w’ prefers m’ to m — By Obs 2

Assumed w’ prefers m to m’



Source: 4simpsons.wordpress.com

Case 2: w’ had proposed to m



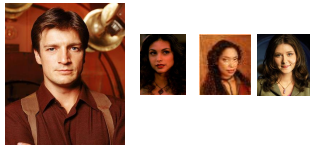
Case 2.1: m had accepted w’ proposal

m is finally engaged to w

Thus, m prefers w to w’

4simpsons.wordpress.com

By Obs 1



Case 2.2: m had rejected w’ proposal

m was engaged to w’’ (prefers w’’ to w’)

By Algo def

m is finally engaged to w (prefers w to w’’)

By Obs 1

m prefers w to w’

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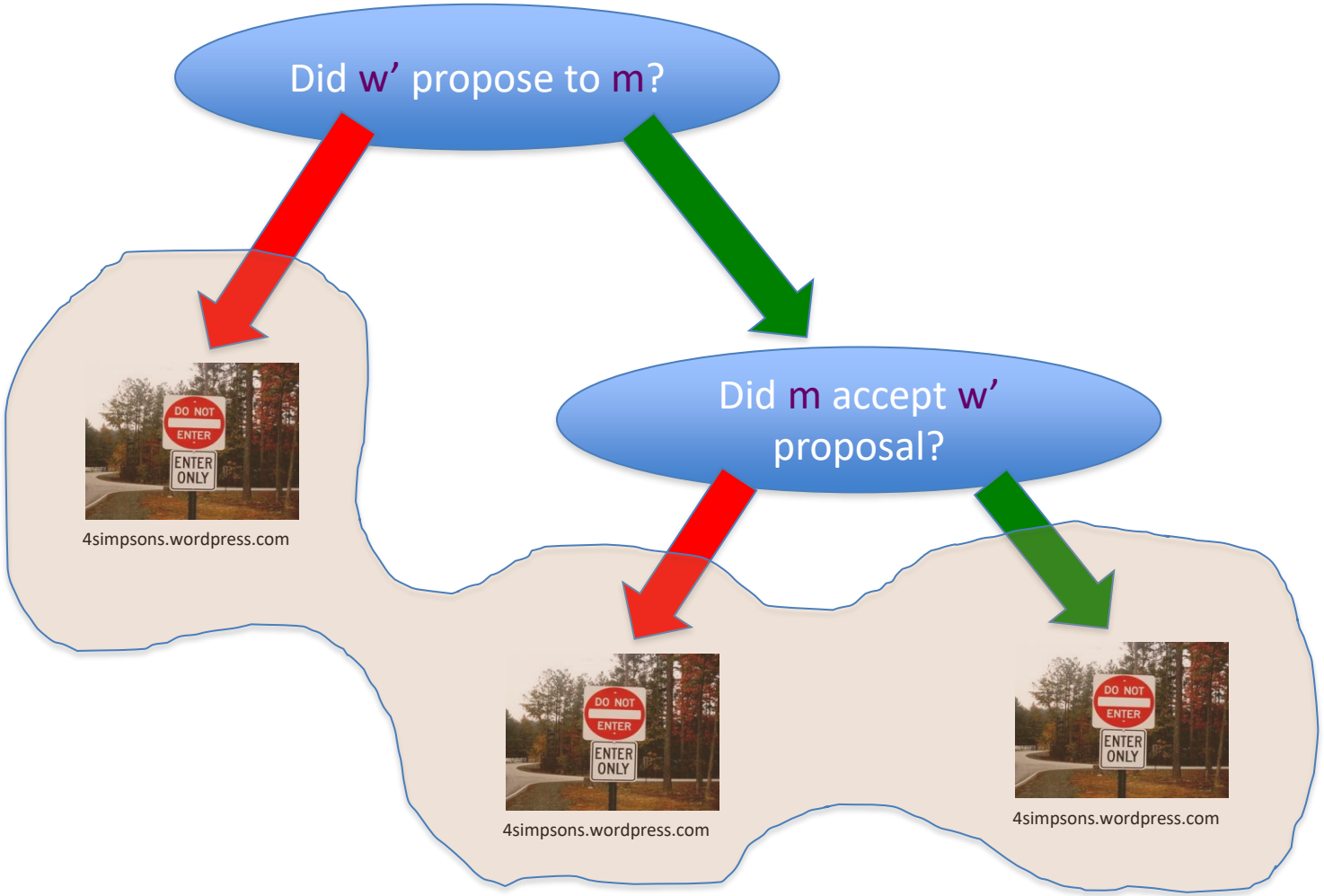
Overall structure of case analysis



Questions?

Extensions

Fairness of the GS algorithm

Different executions of the GS algorithm