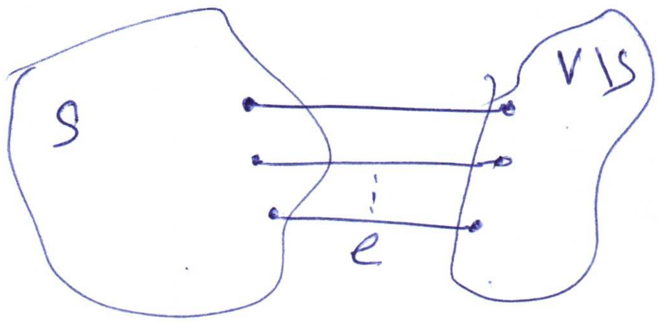
Apr 1

Cut Property Lemma

Assume ALL c_es are distinct.

For all cuts $(S, V \setminus S)$ s.t. $S \neq \emptyset$, $V \setminus S \neq \emptyset$



Consider all "crossing" edges.

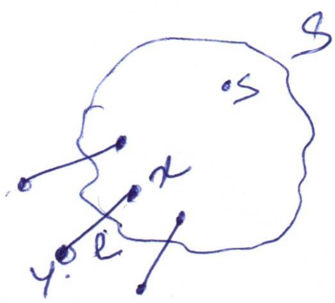
Let e be the cheapest crossing edge.

$\Rightarrow$ e is in ALL MSTs of G.

Assume Cut Property Lemma is true + ALL edge costs (i.e., c_e) are distinct.

THM1: Prim's algo is correct.

Consider the algo when it is about to add e to T.



Goal: show e is the cheapest crossing edge across some cut.

Apply cut property lemma to cut $(S, V \setminus S)$ where S is in prim's algo.

claim 1: e is the cheapest crossing edge

$\Rightarrow$ follows from the def^n of Prim's.

claim 2: $S \neq \emptyset$ (as $x \in S$)

claim 3: $V \setminus S \neq \emptyset \Rightarrow S \neq V$ (as $y \notin S$)

claims 1+2+3 $\Rightarrow$ Prim's algo always adds ~~a~~ safe edge e to S (as e is in all MSTs)

claim 4: At the end of each iteration (S, T) is connected.

Ex.

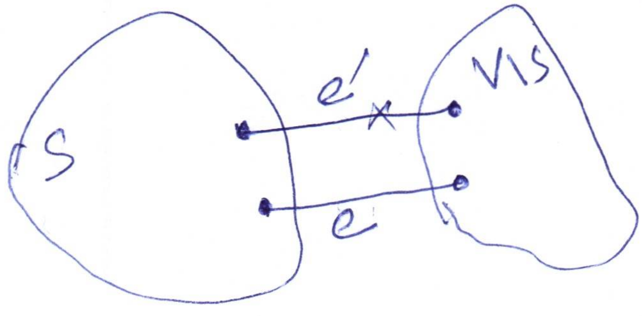
$\Rightarrow$ At the end of execution of prim's, (V, T) is connected.

claim 1+2+3+4 $\Rightarrow$ THM 1

Cut Property Lemma

Pf (idea): By contradiction.

Assume that $\exists$ a cut $(S, V \setminus S)$ and an MST (*) $T = (V, E')$ s.t. the cheapest crossing edge is not in T.



Since T is connected, $\exists$ a crossing edge $e' \in E'$.

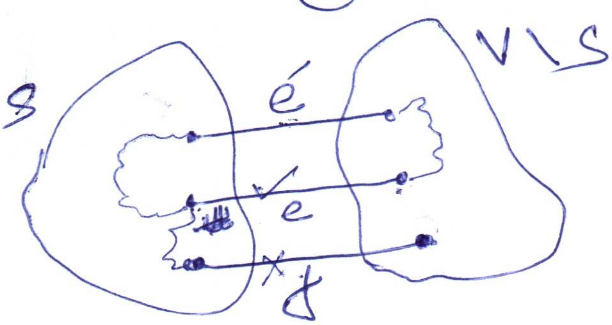
Consider $T' = (V,\ E' \setminus \{e'\} \cup \{e\})$

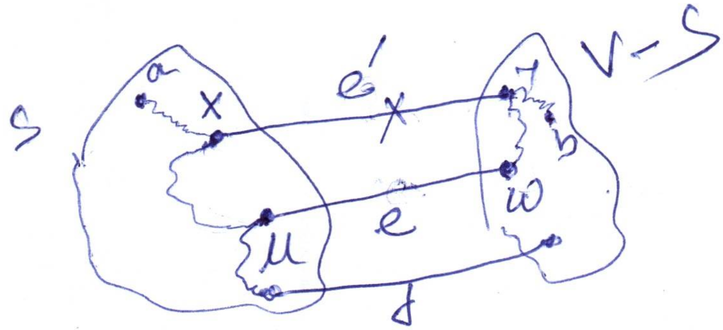
$c(T') < c(T)$

$c(T') = c(T) + c_e - c_{e'}$
$< c(T)$ (as $c_{e'} > c_e$)

$\Rightarrow T$ is not an MST

$\Rightarrow$ contradicts (*)





Since T is connected $\Rightarrow \exists$ u-w path in T

$\Rightarrow \exists$ x-y path s.t. $\exists$ u-x path + $\exists$ w-y path

$\Rightarrow \exists\ e' = (x, y)$

Consider $T' = (V,\ E' - \{e'\} \cup \{e\})$

claim 1: $c(T') < c(T)$ (as $c_{e'} > c_e$)

$c(T') = c(T) - c_{e'} + c_{e}$

claim 2: T' is connected.

Case 1: $\exists$ a-b path that doesn't use e'.

$\Rightarrow$ T' is still connected.

Case 2: $\exists$ a-b path that uses e'.

$\Rightarrow$ use the longer route

$\Rightarrow$ T' is still connected.

claim 1 + claim 2 $\Rightarrow$ cut property lemma.

$\square$