Apr 18

Weighted Interval Scheduling problem

Input : n intervals. ith interval $= (s_i, f_i, v_i)$

Start time (s_i), finish time (f_i), value (v_i)

Output : Instead of outputting an optimal solution O, output $v(O) = \sum_{i \in O} v_i$

Def OPT(j) : value of optimal solution for [j]

$1 \le j \le n$

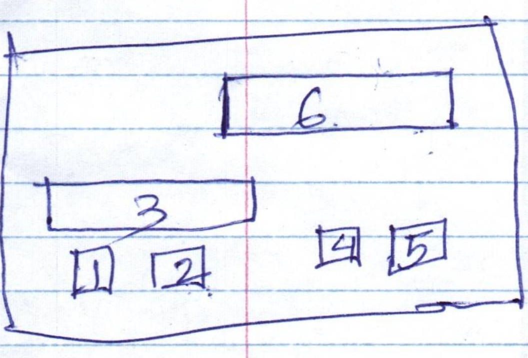
$[j]$: (s_1, f_1, v_1) … (s_j, f_j, v_j)

Goal : OPT(n).

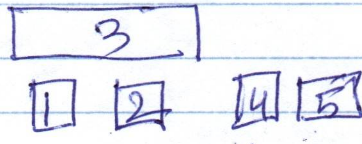
$f_1 \le f_2 \le \cdots \le f_n$

Def let O_j be an optimal solution for [j]

$$v(O_j) = OPT(j) \quad — \quad (*)$$



Case 1 : $j \notin O_j \Rightarrow 6 \notin O_6$



O_6 is an optimal solution for [5]

$\boxed{5}$

$O(6) = OPT(5)$

$\rightarrow OPT(j) = OPT(j-1), \; j \notin O_j$

APP 20

Case 2: $j \in O_j$ $\quad 6 \in O_6$

$O_6 \setminus \{6\}$ is an optimal solution

$\boxed{1}\ \boxed{2}$ for $\{1,2\} = [2]$

$OPT(6) = v_6 + OPT(2)$

$P(6) = 2$

$OPT(j) = v_j + OPT(P(j))$

$P(j)$: largest integer $i < j$ s.t. i & j don't conflict.

$$OPT(j) = \max\{\underbrace{OPT(j-1)}_{j \notin O_j},\ \underbrace{v_j + OPT(P(j))}_{j \in O_j}\} \quad O_j \setminus \{j\}$$

Case 1: $j \notin O_j$

Pf (idea): By contradiction:

claim 1: O_j is also an optimal solution for $[j-1]$

Assume O_j is not an optimal solution for $[j-1]$.

$\exists$ a feasible/valid solution O' for $[j-1]$ s.t. $v(O') > v(O_j)$

Q: Is O' a valid solution for $[j]$.

A: Yes! We know, $OPT(j) \overset{(*)}{=} v(O_j) \overset{\text{claim}}{=} OPT(j-1)$

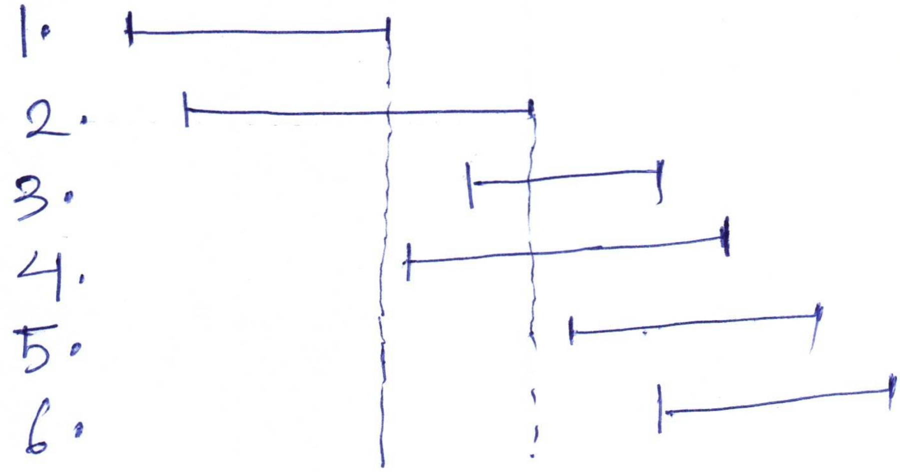
$\hookrightarrow$ contradicts the optimality of O_j as we have another valid solution O' AND $v(O') > v(O_j)$.

= contradicts $(*)$

Case 2: $j \in O_j$

$$P(j) = \begin{cases} \text{largest integer } i < j \text{ s.t. } i \,\&\, j \text{ don't conflict} \\ 0 \quad \text{o/w} \end{cases}$$

$f_1 \le f_2 \le \cdots \le f_n$



$P(1) = 0$
$P(2) = 0$
$P(3) = 1$
$P(4) = 1$
$P(5) = 2$
$P(6) = 2$

Note:
1. $P(j)+1, \ldots, j-1$ conflict with j
2. $1, \ldots, P(j)$ don't conflict with j

$\Rightarrow$ if we pick $j \in O_j$, the remaining subproblem
$1, \ldots, P(j)$

Claim 2: $O_j \setminus \{j\}$ is an optimal solution for $[P(j)]$

$$OPT(j) \overset{*}{=} v(O_j) = v_j + v(O_j \setminus \{j\})$$
$$\overset{\text{claim2}}{=} v_j + OPT(P(j))$$

Pf (idea): By contradiction.
$\exists$ a feasible solution O' for $[P(j)]$ s.t. $v(O') > v(O_j \setminus \{j\})$

Note: $O' \cup \{j\}$ is a valid solution for $[j]$

$$v(O' \cup \{j\}) = v(O') + v_j$$
$$> v(O_j \setminus \{j\}) + v_j$$
$$= v(O_j) - \cancel{v_j} + \cancel{v_j}$$
$$= v(O_j)$$
$$\Rightarrow \text{contradicts } (*)$$

Ex1: We can compute $P(1) \dots P(n)$ in $O(n \log n)$ time.

Ex2: Any algorithm to compute $P(1) \dots P(n)$ needs $\Omega(n \log n)$ time/comparison.