Shortest path problem

Input: Directed graph $G = (V, E)$

Start vertex s

"lengths" $l_e \geq 0 \ \forall e \in E$

↑ integers

OUTPUT: A shortest s-t path P $\forall t \in V$

↑ w.r.t. the length of P

$$l(P) = \sum_{e \in P} l_e$$

Simpler Version: Output $d(t)$ $\forall t \in V$

$d(t)$: distance of t from s

↑ in term of length

$\Rightarrow$ length of the shortest s-t path

Special Case: $l_e = 1 \ \forall e \in E \equiv$ HW3 Q3 (run BFS from s and layer # $t = d(t)$)

Q: $l_e = L$ (for some $L \geq 1$)

e.g. $L = 100$

$l_e = 100 \ \forall e \in E$

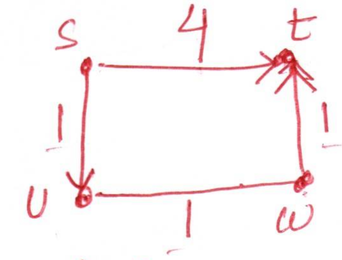
A: Reduce this case to HW3 Q3

How: Ignore L, assume $l_e = 1$ $\forall e \in E$

3/18 General case: $l_e \geq 0, \forall e \in E$ [l_e can have different lengths]

Algo Idea: Reduce the general case to $l'_e = 1$ and run HW3 Q3 algo.

Idea 1: Ignore $l_e \Rightarrow$ assume $l_e = 1 \; \forall e \in E$ and run HW3 Q3 algo.

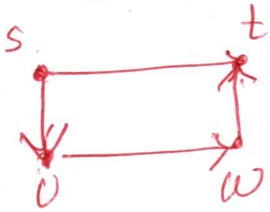


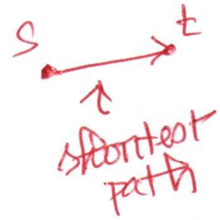
$d(t) = 3$

P: s, u, w, t

$l(P) = 3$

Idea 1 → ignore l_e $\Rightarrow$ assume $l_e = 1 \; \forall e \in E$

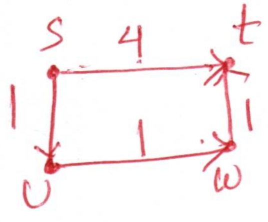




Report $d(t) = 4$ ✗

* BFS works only if the edge lengths are the same.

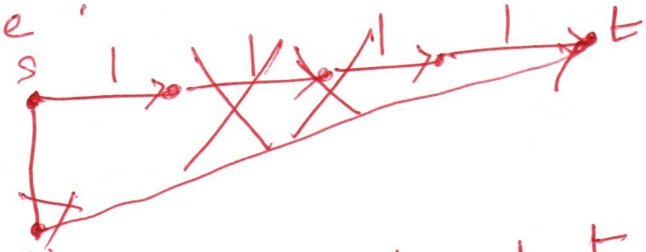
Idea 2: Replace an edge e of length $l_e > 1$, by a path of length l_e.

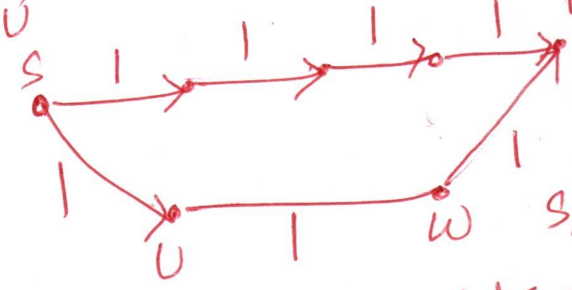


G

$d(t) = 3$

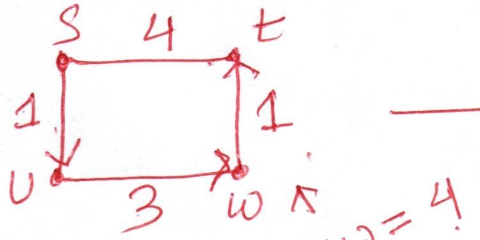
Idea 2 →





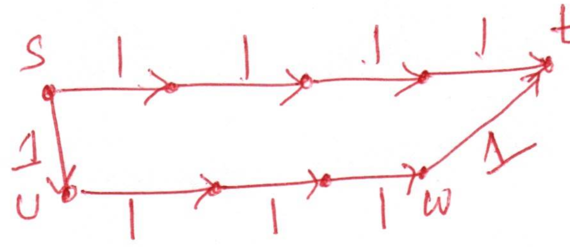
Run BFS ⇓ shortest path: s, u, w, t

$d(t) = 3$ ⇐ G'



$d(t) = 4$

Idea2



$G = (V, E)$

BFS Runtime : $O(m+n)$

$n = |V|$ $m = |E|$

G'

$\Rightarrow$ Run BFS

$\Rightarrow$ shortest path : s, t

$d(t) : 4$

Runtime Analysis : $G' = (V', E')$

Connectness of Idea 2 :

claim : shortest path in G

$\Rightarrow$ equivalent to shortest path in G'

$\Rightarrow$ proof of claim + connectness of HW3 Q3 algo.

$G' = (V', E')$ $|V'| = n'$

$|E'| = m'$

$l_{max} = \max_{e \in E} l_e$

$\Rightarrow n' \leq l_{max}(n+m)$

Runtime : $O(n' + m')$ $\Rightarrow m' \leq l_{max}(m+n)$

$= O(l_{max}(n+m))$

if $l_{max} = 5 \Rightarrow O(5(m+n))$

$\quad '' = n^{100} \Rightarrow O(n^{100}(m+n))$

The graph G is in adj. list format:

Space: $O(m+n)$

<u>RAM model:</u> Unit of space is ~~a~~ register.

If you have n input ~~to~~ items

$\Rightarrow$ how many bits is a register?

$\Rightarrow \log n$ bits register

w bits
$\Rightarrow 2^w$ items
$2^w \geq n$
$w \geq \log_2 n$

$\Rightarrow$ How many ~~to~~ bit registers for n^{100}?

A: $2^w \geq n^{100}$

$w \geq 100 (\log n)$

$\Rightarrow 100$ registers

$\Rightarrow O(1)$ "

$\Rightarrow$ runtime: $O(n^{100}(m+n))$