Proof of Correctness of Greedy Algorithm (Interval Scheduling problem)

* $f(1) \le f(2) \le \cdots \le f(n)$

t_0	0	t_2
t_1	1	t_1
t_2	2	t_0
t_3	3	t_3

$f(t_2) \le f(t_1) \le f(t_0) \le f(t_3)$

0. $R \leftarrow [n]$
1. $S \leftarrow \emptyset$
2. While $R \neq \emptyset$
   - (2.1) pick $i \in R$ with smallest index
   - (2.2) add i to S
   - (2.3) remove $\forall j \in R$ s.t. j conflicts with i
3. $S^* \leftarrow S$

   return S^*

THM1: For all inputs, S^* is an optimal solution.

$\Rightarrow$ for all inputs and all possible valid schedules S^* has the max # intervals.

Let O be an optimal solution.

Idea prove/show $S^* = O$ ✗ as there are multiple optimal solutions possible.

THM2: $|S^*| = |O|$

Notations: $S^* = \{i_1, i_2, \ldots, i_k\}$ $f(i_1) \le f(i_2) \le \cdots \le f(i_k)$

$O = \{j_1, j_2, \ldots, j_m\}$ $f(j_1) \le f(j_2) \le \cdots \le f(j_m)$

THM2': $k = m$

Claim 1: $K \le m$ (as O is an optimal solution)

Lemma 1: (Greedy stays ahead)

$\forall\ 1 \le l \le K, \quad f(i_l) \le f(j_l)$

[Assume Lemma 1 is true] ..

proof (idea) of THM 2: proof by contradiction.

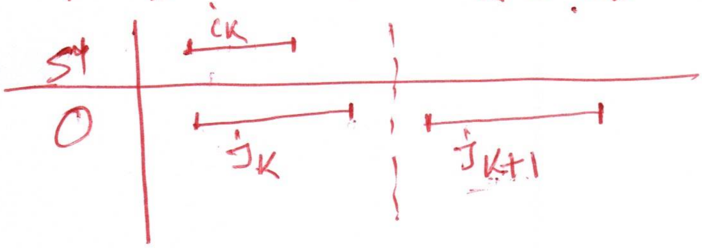
Assume $K \neq m$

$\xRightarrow{\text{by claim 1}} K < m$

$\Rightarrow m \geq K+1$

$\Rightarrow j_{K+1} \in O$

By Lemma 1, $f(i_K) \le f(j_K)$



Consider the algo right after i_K has been added to S.

* [Note: i_K is the last interval added to S^*]

$\Rightarrow j_{K+1} \in R \Rightarrow$ [j_{K+1} is not in conflict with $i_K, i_{K-1}, i_{K-2}, \ldots, i_1$]

$\Rightarrow R \neq \emptyset$

$\Rightarrow$ Greedy algo. cannot terminate

$\Rightarrow$ Contradicts

Proof (idea) of Lemma 1 : $(\forall\, 1 \le l \le k,\ f(i_l) \le f(j_l))$

proof by induction on l.

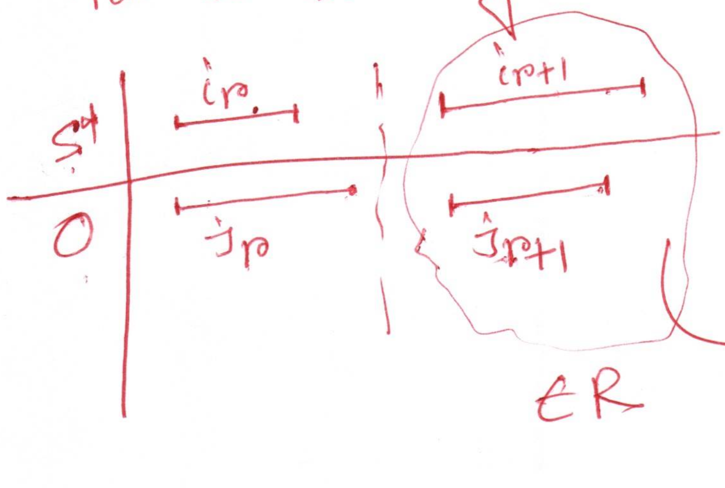
Base Case : $l = 1$ $\quad f(i_1) \le f(j_1)$ $\rightarrow f(j_1)$

I.H. : For some $r \geq 1$, Assume

$\forall\, 1 \le l \le r,\ f(i_r) \le f(j_r)$.

I.S. : $f(i_{r+1}) \le f(j_{r+1})$

For the sake of contradiction, $f(i_{r+1}) > f(j_{r+1})$



Consider the algo right after i_r has been added to S.

$i_{r+1} \in R$

$j_{r+1} \in R$

$\Rightarrow$ algo. cannot pick i_{r+1}

$\Rightarrow$ contradicts $i_{r+1} \in S^*$