Satisfaction of a Quantified WFF

In case my explanation in lecture of the definition of satisfaction for a quantified wff wasn't clear, here it is at greater length (for a finite case, which may make it easier to understand, and can be done "wolog", i.e. "without loss of generality").

Suppose our domain, D, contains the following items:

$$d_1, \ldots, d_{200}$$

Suppose our language has these variables:

$$x_1, \ldots, x_{100}$$

The definition of satisfaction for a universally quantified wff is:

$$\Im, \mu \models \forall v.\alpha \text{ iff } \Im, \mu' \models \alpha \text{ for all } \mu' \text{ that differ from } \mu \text{ at most on } v.$$

E.g., suppose v is the variable: x_{50}
and α is the wff: "${x_{50}}^2 = x_{50} * x_{50}$"

Consider the set of all variables: $\{x_1, \ldots, x_{50}, \ldots, x_{100}\}$

Consider what set of members of D we get by applying a variable assignment μ to all the members of this set:

$$\{\mu(x_1), \ldots, \mu(x_{50}), \ldots, \mu(x_{100})\} \subseteq D$$

Let's say that this set = $\{d_1, \ldots, d_{50}, \ldots, d_{100}\}$.

Now consider all μ' that differ from μ at most on x_{50}. Suppose they are $\mu_1, \ldots, \mu_{300}$.

(It may be that $\mu_{301}, \ldots, \mu_{1000}$ differ from μ on other variables, too, so we don't consider these at all!)

Note that our μ above is one of these 300 μ' (because all the μ' differ **at most** on x_{50}, meaning that they might not differ on x_{50} at all). For convenience, let's say that our μ above is μ_1.

So consider what all these μ' do to the set of all variables; they map them into sets of elements of D as follows:

$$\begin{array}{rcl}\mu_1 & : & \{\mu_1(x_1),\ldots,\mu_1(x_{50}),\ldots,\mu_1(x_{100})\} \\ & & . \\ & & . \\ & & . \\ \mu_{300} & : & \{\mu_{300}(x_1),\ldots,\mu_{300}(x_{50}),\ldots,\mu_{300}(x_{100})\}\end{array}$$

Note, in particular, that μ_1 gives us: $\{d_1,\ldots,d_{50},\ldots,d_{100}\}$. What about the others? Well, each differs from μ (or μ_1) at most on what it does to x_{50}, so we get the following:

$$\begin{array}{rcl}\mu_1 & : & \{d_1,\ldots,d_{49},d_{50},d_{51},\ldots,d_{100}\} \\ \mu_2 & : & \{d_1,\ldots,d_{49},d_1,d_{51},\ldots,d_{100}\} \\ & & . \\ & & . \\ & & . \\ \mu_{300} & : & \{d_1,\ldots,d_{49},d_{77},d_{51},\ldots,d_{100}\}\end{array}$$

I.e., they are all alike on the first 49 variables and on the last 50 variables, but they can vary wildly on good old x_{50}.

Now, according to the definition, if all of these μ' satisfy "${x_{50}}^2 = x_{50} * x_{50}$", then our original μ will satisfy "$\forall x_{50}[{x_{50}}^2 = x_{50} * x_{50}]$".

But to decide if a given μ' satisfies "${x_{50}}^2 = x_{50} * x_{50}$", we can ignore what it does to everything except x_{50}.

(We can do this for 2 reasons: First, they **don't** differ on the other variables. Second, our wff only contains that one variable, so that's the only one we care about.)

On x_{50}, each μ' differs. Does each one satisfy "${x_{50}}^2 = x_{50} * x_{50}$"? Sure! Here's the proof:

$$\begin{array}{c}{d_{50}}^2 = d_{50} * d_{50} \\ {d_1}^2 = d_1 * d_1 \\ \ldots \\ {d_{77}}^2 = d_{77} * d_{77}\end{array}$$

So: all the μ' that differ from mu at most on x_{50} satisfy "${x_{50}}^2 = x_{50} * x_{50}$", so μ satisfies "$\forall x_{50}[{x_{50}}^2 = x_{50} * x_{50}]$".

And similarly, "mutatis mutandis" (as mathematicians say; i.e., changing what needs to be changed), for the existential quantifier.

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http://www.cse.buffalo.edu/∼rapaport/563S05/satisfaction-2005-03-04.tex