Notes on the Unification Algorithm

William J. Rapaport

(Based on material from Chang & Lee 1973.)

Last Update: 28 September 2003

Note: ***NEW*** or ***UPDATED*** material is highlighted

1 Background Definitions and Examples

Definition

Let t be a term (constant, variable, or functional term).
Let v be a variable.
Then v/t **is a binding** (or: **is a variable assignment**) $=_{df}$ t is substituted for v (or: v is assigned, or bound to, t).

- Other notations: t/v, $v := t$, $v \leftarrow t$

Definition

A **substitution** is$_{df}$ a finite set $\theta = \{v_1/t_1, \ldots, v_n/t_n\}$ of bindings such that, for all i, j:

1. $t_i \neq v_i$, and
2. if $v_i/t_i \in \theta$ and $v_j/t_j \in \theta$, then $v_i \neq v_j$
   (i.e., don't substitute 2 different terms for the same variable).

Definition

Let $\theta = \{v_1/t_1, \ldots, v_n/t_n\}$ be a substitution.
Let E be an expression (wff or term).
Then:

1. $E\theta =_{df}$ the expression obtained from E by *simultaneously* (i.e., *in parallel*) replacing each v_i in E by t_i.
2. $E\theta$ is$_{df}$ an **instance** of E.
3. If $W = \{E_1, \ldots, E_n\}$, then $W\theta =_{df} \{E_1\theta, \ldots, E_n\theta\}$.

- Other notation: $E^{v_1,\ldots,v_n}_{t_1,\ldots,t_n}$

Example

Suppose $\theta = \{x/a,\ y/f(b),\ z/c\}$,
and $E = P(x,y,z)$.
Then $E\theta = E\{x/a,\ y/f(b),\ z/c\} = P(a,\ f(b),\ c)$.

Definition

Let $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$
and $\sigma = \{y_1/u_1, \ldots, y_m/u_m\}$ be substitutions.
Then **the composition of** θ **with** σ

$$\theta \circ \sigma$$

$=_{df}$ $\{x_1/t_1\sigma,\ \ldots,\ x_n/t_n\sigma,\ y_1/u_1,\ \ldots,\ y_m/u_m\}$
UPDATE $- \{x_j/t_j\sigma \ :\ t_j\sigma = x_j\}$ // eliminate "redundant substitutions" x_j/x_j
$-\{y_j/u_i \ :\ y_i \in \{x_1, \ldots, x_n\}\ \}$ // eliminate clashes, but let θ-as-modified-by-σ trump σ.

- **Note:**
  $\circ$ is *not* commutative! To see why, let $\theta = \{x/f(y),\ y/z\}$ and let $\sigma = \{x/a,\ y/b,\ z/y\}$.
  Then compute $\theta \circ \sigma$ and $\sigma \circ \theta$.

Definition

Let θ be a substitution.
Let $E_1, \ldots, E_n$ be expressions.
Then θ **is a unifier of** $\{E_1, \ldots, E_n\}$ (and $E_1, \ldots, E_n$ **are unifiable**) $=_{df}$ $E_1\theta = \ldots = E_k\theta$
(i.e., iff $\{E_1\theta, \ldots, E_n\theta\}$ is a singleton).

- I.e., $E_i\theta$ is a "common instance" of $E_1, \ldots, E_n$.

Example

$\theta = \{x/a,\ y/f(b)\}$ unifies $\{P(a,y),\ P(x,\ f(b))\}$.
The common instance is: $P(a,\ f(b))$.

Definition

Let θ be a unifier of $\{E_1, \ldots, E_n\}$.
Then θ **is a [NOT 'the'!] most general unifier (MGU) of** $\{E_1, \ldots, E_n\}$ $=_{df}$
$(\forall$ (other) unifier τ of $\{E_1, \ldots, E_n\})(\exists$ substitution $\sigma)[\tau = \theta \circ \sigma]$

- I.e., any (other) unifier τ is the composition of θ with some substitution σ.

2 The Unification Algorithm

Algorithm Unification; // Chang & Lee
input non-empty set of expressions W;
output θ = MGU(W) or failure;
begin

1. $k := 0; W_0 := W; \theta_0 := \{\ \}$;

2. **if** W_k is singleton **then return** θ_k
   else DS_k := Disagreement_Set(W_k);

3. **if** ($\exists$ var v_k, term $t_k \in DS_k)[v_k$ does not occur in t_k]
   // "occurs-check"
   then begin
   $\theta_{k+1} := \theta_k \circ \{v_k/t_k\}$;
   $W_{k+1} := W_k\{v_k/t_k\}$;
   // i.e., apply substitution to each member of W_k
   // N.B.: $W_{k+1} = W\theta_{k+1}$
   $k := k+1$;
   goto 2
   end
   else return failure

end.
For an example, see <http://www.cse.buffalo.edu/∼rapaport/663/F03/unification.eg.pdf>

3 Reference

1. Chang, Chin-Liang & Lee, Richard Char-Tung (1973), *Symbolic Logic and Mechanical Theorem Proving* (Academic Press).

rapaport@cse.buffalo.edu
http://www.cse.buffalo.edu/∼rapaport/563/unification.2003.09.16.pdf