PAPER FOLDING AND CONVERGENT SEQUENCES

Paper folding can help in understanding some infinite sequences and in finding their limits. A simple physical model useful at all levels of ability is presented and infinite sequences of interest to senior high school students are explored.

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THERE are many well-known physical representations of convergent sequences. A standard example is to traverse the length of a room by walking half the remaining distance each time a "step" is taken. One sequence corresponding to this model ($0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$) has 1 as its obvious limit.

There is a sequence (see table 1) that also has a simple and interesting physical model but whose limit is not so obvious. And precisely because of these facts, the sequence and its physical representation can be used as the basis and motivation for several interesting lessons.

The physical model requires the student (or the teacher) to fold a strip of paper according to the directions given in the next section. One lesson that can be based on this exercise involves discovery of a mathematical description of the resulting sequence and finding the limit of this sequence. This is done in the section called "Mathematical Description of the Sequence." In the last section, "Other Related Lessons," some other lessons that can be built around the physical model are suggested.

Obtaining a Sequence by Folding

Take a strip of adding-machine tape at least twelve inches long. (Any size piece of paper that is suitable for folding will do, but at least one of its edges must be straight). Label the left edge A and the right edge B (see fig. 1a).

The sequence of folds is as follows:

1. Fold B to the *left* to coincide with A (see fig. 1b); call the crease that is created by this fold, C (see fig. 1c).
2. Without unfolding the paper, fold B to the *right* to coincide with C, forming a second crease, D (see fig. 1d). Just as a check, the paper (when viewed from the side) should now look like fig. 1e).
3. Without unfolding, fold B to the *left* to coincide with D, creating crease E (top view in fig. 1f, side view in fig. 1g).
4. Continue as before, successively folding B to the right, left, and so on, so that at the end of each folding operation, B coincides with the crease made last.

After three more folds, for example, the side view should look like figure 1h. (Making the folds is actually quite simple, certainly much simpler than the verbal description.)

An interesting question comes to mind immediately: Where will edge B appear if one could continue folding indefinitely?

Some students might not be convinced that this question is meaningful (unlike the room-traversing example where most students are certain that the other side of the room would eventually be reached). One way to clarify the question is to unfold the paper so that it resembles figure 1a again, except that now it is creased. Make a cut with scissors from B to fold

The author wishes to thank Professor Stanley Taback for his helpful comments on an earlier draft of this paper.

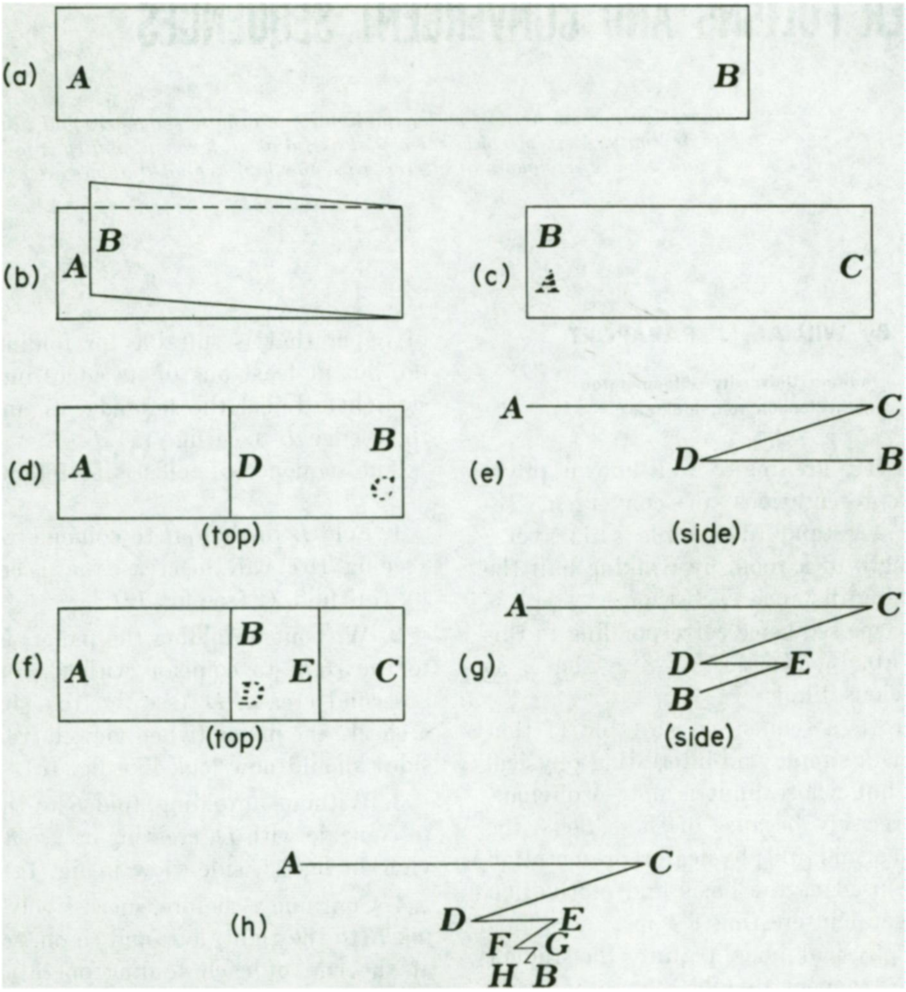

Fig. 1

C and then up to the top edge (see fig. 2), thus cutting off one-fourth of the paper. Now refold the paper as before, this time marking a dot after each fold is made in order to show where B lands (see fig. 3).

If the dots are thought of as points on a number line, you should be able to see how they begin to cluster around one point. (This can be made even clearer by using a longer piece of paper, thus making it possible to make more folds.) It is the cluster, or limit, point that we desire to find. You might guess, by inspection, that the limit point is a certain, very simple, fraction of the way from A to B. Let us now find out exactly.

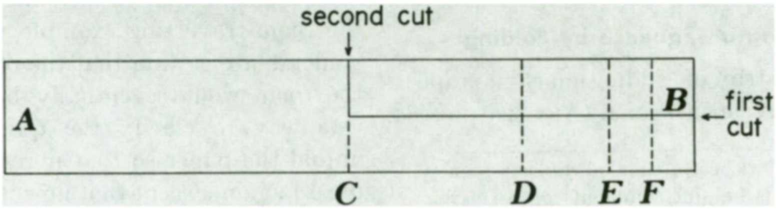

Fig. 2

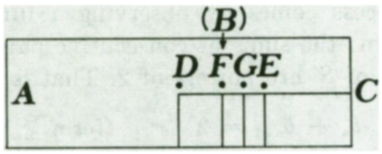

Fig. 3

Mathematical Description of the Sequence

Think of the top edge of the paper as a number line, with 0 at edge A and 1 at edge B. We have already mapped the sequence of folds onto a sequence of dots or points. We now want to map this point sequence onto a sequence of numbers so that we can employ numerical techniques to answer our question. The trick is to do this in an efficient way. Allowing each student in a laboratory situation to do this in his or her own way is advisable. Alternatively, some class time could be spent on deciding precisely what mapping to use. I suggest the following.

After fold 1, B is at 0 on the number line. After fold 2, B is at $\frac{1}{2}$, since it then coincides with crease C (which was the result of folding the paper in half). After fold 3, B coincides with crease D, which was obtained by folding the halved paper in half; B is now at $\frac{1}{4}$. Although the fourth fold halves again, the result is not $\frac{1}{8}$, but $\frac{3}{8}$. The reason for this may be seen by realizing that the exercise is a physical model for *averaging*: after the fourth fold, B is in the *middle* of its last two positions ($\frac{1}{4}$ and $\frac{1}{2}$). Folding, that is, averaging, a fifth time yields $\frac{5}{16}$. Table 1 may be derived.

TABLE 1

Fold	Position of B
1	0
2	1/2
3	1/4
4	3/8
5	5/16
6	11/32
7	21/64
8	43/128
9	85/256
10	171/512
11	341/1024
.	.
.	.
.	.

We thus obtain the following sequence (of positions of B):

$$S\colon 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \dots .$$

Our question may now be phrased more precisely: What is the limit of sequence S?

In order to answer this question, it is necessary to find a formula for arbitrary terms of S. Clearly, the denominators of the terms are powers of 2. In fact, for fold n, the denominator (of the fraction representing the position of B after the nth fold) is 2^{n-1}. Let the first term (i.e., 0) in sequence S be called a_1, the second term (i.e., $\frac{1}{2}$) a_2, and so on, so that we may say that the denominator of a_n is 2^{n-1} (for $n \geq 1$; it is assumed throughout that n is a natural number). Now the numerator of a_n for each n is needed.

It is at this point that the students' problem-solving abilities are put to the test, for although there are many patterns to be found in this sequence, a pattern that will be useful for our purposes can prove to be quite elusive. One method (out of many) begins by expressing S recursively (where a_1 and a_2 are given) and then finding an equivalent formula in which a_n depends only on n (i.e., a formula that does not require knowing in advance any terms of S).

We have already seen that each term is the average of the two preceding terms. (Note that before folding, i.e., at "fold" 0, B is located at 1. Question: Where is B at "fold" -1?) Thus,

$$(1) \qquad \begin{aligned} a_1 &= 0 \\ a_2 &= \tfrac{1}{2} \\ a_{n+2} &= \frac{a_n + a_{n+1}}{2} \qquad (\text{if } n \geq 1). \end{aligned}$$

Formula (1) is a recursive formula (with two initial conditions) that generates S. It allows us to imagine folding as many times as we wish, eliminating the practical limitations of the thickness and length of the paper. The problem, though, is to find

$\lim_{n\to\infty} a_n$, and, as interesting as (1) may be, it does not help much.

Recall that a formula is needed for the numerator of a_n for each n. That is, a formula is wanted for an arbitrary term of the sequence S' of numerators:

$$S': \ 0, 1, 1, 3, 5, 11, \ldots .$$

Call each term of this new sequence b_n (for $n \geq 1$). Thus, $a_n = b_n/2^{n-1}$ (for $n \geq 1$). Next is the derivation of a simple recursive formula for any term of S', wherein each term depends on the two previous terms. The determination of this formula is a nice short exercise by itself. The result follows:

$$\begin{aligned} &b_1 = 0 \\ (2) \qquad &b_2 = 1 \\ &b_{n+2} = 2b_n + b_{n+1} \qquad (\text{if } n \geq 1). \end{aligned}$$

However, this is not just what was wanted.

There is a more interesting line of attack. Is there any property of either sequence, S or S', that reflects the physical fact that the folds alternate from right to left? There are at least two such properties.

First, observe that each term of S' (after the first) is either one more than or one less than twice the preceding term; that is, $b_{n+1} = 2b_n \pm 1$ (for $n \geq 1$). One can then verify (or discover) formula (3):

$$\begin{aligned} (3) \qquad &b_1 = 0 \\ &b_{n+1} = 2b_n + (-1)^{n-1} \quad (\text{if } n \geq 1). \end{aligned}$$

Here, the alternation in folding direction is reflected in the alternating parity of the terms of the sequence $(1, -1, 1, -1, \ldots)$ generated by $(-1)^{n-1}$. Formula (3) is halfway to the general formula; it represents an improvement over (2) in that it has only *one* initial condition. In order to eliminate the need for *any* such initial condition, it is necessary to express the nth term of S' (or S) in terms of n only.

Success comes by observing a further pattern: the sums of consecutive pairs of terms of S' are powers of 2. That is,

$$(4) \qquad b_n + b_{n+1} = 2^{n-1} \qquad (\text{for } n \geq 1).$$

Since (4) is equivalent to

$$(5) \qquad b_{n+1} = 2^{n-1} - b_n \qquad (\text{for } n \geq 1),$$

the right-hand sides of (3) and (5) may be set equal to each other in order to find the desired formula ($n \geq 1$, throughout).

$$2b_n + (-1)^{n-1} = 2^{n-1} - b_n$$

$$\therefore \quad 3b_n = 2^{n-1} - (-1)^{n-1}$$

$$(6) \qquad \therefore \quad b_n = \frac{2^{n-1} - (-1)^{n-1}}{3}$$

$$(7) \qquad \therefore \quad a_n = \frac{2^{n-1} - (-1)^{n-1}}{3 \cdot 2^{n-1}}.$$

With (7) we have reached our goal. It now remains to find $\lim_{n\to\infty} a_n$. This may be done as follows:

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2^{n-1} - (-1)^{n-1}}{3 \cdot 2^{n-1}} \\ &= \frac{1}{3} \cdot \lim_{n\to\infty} \frac{2^{n-1} - (-1)^{n-1}}{2^{n-1}} \\ &= \frac{1}{3}\left(\lim_{n\to\infty} \frac{2^{n-1}}{2^{n-1}} - \lim_{n\to\infty} \frac{(-1)^{n-1}}{2^{n-1}}\right) \\ &= \frac{1}{3}\left(1 - \lim_{n\to\infty} (-\tfrac{1}{2})^{n-1}\right). \end{aligned}$$

But $|-1/2| < 1$; therefore, $\lim_{n\to\infty} (-1/2)^{n-1} = \lim_{n\to\infty} (-1/2)^n = 0$ (cf. Walter Rudin, *Principles of Mathematical Analysis,* theorem 3.20). Hence, $\lim_{n\to\infty} a_n = 1/3$, as the reader may have conjectured. (An interesting point to consider is this: Where did the 3 come from, when it was powers of 2 that seemed to play such an important role? *Hint:* Look at (6).)

A different method of solution arises from the aspect of S' that reflects the alternating property of the folding rule. Two subsequences can be derived from S'—one for even values of n and one for odd values of n (see table 2).

Clearly, the odd values correspond to

TABLE 2

Odd		Even	
n	b_n	n	b_n
1	0	2	1
3	1	4	3
5	5	6	11
7	21	8	43
9	85	10	171
.	.	.	.
.	.	.	.
.	.	.	.

folds to the *left* and the even values to folds to the *right*. The student should discover (or be led to see) that the pair-wise differences of the terms of the odd subsequence are 1, 4, 16, 64, . . . , and those for the even subsequence are 2, 8, 32, 128, ... That is, for the *odd* subsequence, the pair-wise differences are $2^0, 2^2, 2^4, \ldots, 2^{2k}, \ldots$ $(k \geq 0)$, whereas for the *even* subsequence, the pair-wise differences are $2^1, 2^3, 2^5, \ldots, 2^{2k+1}, \ldots$ $(k \geq 0)$.

The students should then find (following the methods suggested above) formulas for these subsequences, calculate the limits for each subsequence, and see that each limit is ⅓. This alternative approach affords a good example of the fact that if a sequence converges to a limit L, then all of its subsequences also converge to L.

Other Related Lessons

One of the nicer aspects of this exercise is that it can be used at many levels of ability and for many different purposes.

At an elementary level, all the numerical manipulations can be ignored and the emphasis placed on making the folds, in order to give the students an intuitive idea of limits that differs from the more familiar example mentioned at the beginning.

A lesson could also be developed for a unit on measuring. For such a lesson, a twelve-inch length of adding-machine tape and a twelve-inch ruler for each student (or a thirty-six-inch length and a yardstick for a larger group) would be ideal. The student could then measure, after each fold, the distance of each crease from edge A, getting a sequence whose limit is 4″ (or 12″ for the 36″ length). This would not only afford practice in measurement, but also give the students a good feel for the limiting process. (One warning for metric enthusiasts: ⅓ of 12″ is a very "clean" 4″, but ⅓ of 10 cm. might prove impractical. Try a 15 cm. length or some other multiple of 3; after all, a teacher's materials must be prepared in advance just as much as a magician's!)

For a class studying fractions, this exercise could be used to give the students practice in working with fractions, using formula (1) to derive terms in the sequence. Moreover, by changing each fraction into its decimal equivalent, the students not only will have practice in division, but also will be able to "see" the terms of the sequence approach .333 . . . , both from "above" and from "below." And, of course, it may serve as a motivating exercise for a unit on averaging.

Addendum to “Paper Folding and Convergent Sequences”

The method illustrated in the previous paper uses the techniques of continuous mathematics, in particular, limits.

The problem posed in that paper can also be solved using the techniques of discrete mathematics, in particular, recurrence relations.

That technique is described in the following extracts from my Lecture Notes on Discrete Math (November 2010).

Please ignore the partial notes on other topics at the beginning and at the end of the document. The full set of lecture notes for the course is online at https://cse.buffalo.edu/~rapaport/191/

l. $\exists x P(x)$, where x is a variable, is WF by rec case (vii), because $P(x)$ is a WF propositional function by the base case.

5.

a. Question:

- "$\oplus$" is the symbol for exclusive disjunction.
  Let A,B be WFFs of FOL.
  Is $(A \oplus B)$ a WFF of FOL?

b. Answer:

- Not according to the above definition.
- But surely it "should be" considered as a WFF of FOL, right?
- Here's how we can use it without changing the definition of WFF:
  - We can *define* the symbol "$\oplus$" as follows:

    Let A,B be any 2 WFFs of FOL.
    Then let $(A \oplus B)$ be an **abbreviation** for:

    $$((A \vee B) \wedge \neg(A \wedge B))$$

  - In other words,
    any time that we use $(A \oplus B)$,
    we would just be being lazy,
    and we should really write out the full definition,
    which *is* a WFF of FOL.

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§§7.1–7.2: Recurrence Relations

I. **Recursive Def** (of a sequence $\{a_n\}$):

Let a_i be terms of the sequence.
Let C_i be constants. Let h be a function.

Base Cases (or: **Initial Conditions**):

$a_0 = C_0$
…
$a_{n-1} = C_{n-1}$

Recursive Case (or: **Recurrence Relation**):

$a_n = h(a_0, \ldots, a_{n-1})$

II.

A. Normally, $a_n = f(n)$ "explicitly";
i.e.)

the nth term of the sequence would be defined directly in terms of n

B. Questions:

1. What's the relation between f & h?
2. Can we compute f, given h?

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§§7.1–7.2: Recurrence Relations

I. **Recurrence Relations:**

A. A sequence can be defined in two different ways:

1. *non*-recursively ("explicitly"),
   in terms of its *current I/P*:

   $a_n = f(n)$

2. recursively,
   by giving initial conditions (first few terms)

   $a_0 = C_0, \ldots, a_{n-1} = C_{n-1}$

   & a recurrence relation that defines the sequence in terms of its *previous O/P*:

   $a_n = h(a_0 = C_0, \ldots, a_{n-1})$

B. Question: Given initial conditions & recurrence relation $a_n = h(a_0 = C_0, \ldots, a_{n-1})$,
(how) can we compute the explicit formula f(n)?

- This is called "solving" the recurrence relation.

C. E.g.)

1. initial conditions:

   $a_0 = C_0$
   $a_1 = C_1$

   recurrence relation:

   $a_n = 3a_{n-1} - 2a_{n-2}, \forall\ n \geq 2$

2. Lots of different sequences share this pattern,
differing only in their initial conditions

    a. *Given* the initial conditions,
    we can *compute* the nth term ($n \geq 2$)
    *without knowing what the function **does** to its I/P!*

    b. We compute it on the basis of what it **did**:
    we compute it on the basis of what its previous O/P was!

    c. Here are some examples:
    (initial conditions are in the first 2 rows;
    last row shows "explicit" "solution",
    i.e.) def in terms of I/P)

a_0	0	0	1	1	1	2	2
a_1	0	1	0	1	2	1	2
a_2	0	3	–2	1	4	–1	2
a_3	0	7	–6	1	8	–5	2
a_4	0	15	–14	1	16	–13	2
a_5	0	…	–30	1	…	…	…
…	…	…	…	…	…	…	…
a_n	0	$2^n–1$	$2–2^n$	1	2^n	$3–2^n$	2

D. What good are recurrence relations?

1. They describe similar patterns of *growth*,
based on differing *initial conditions* or "seeds"

2. E.g.) compound interest:

    - 2 people deposit different amounts of $ in same bank;
    ∴ same recurrence relation computes their interest;

      But the actual interest depends on their initial deposit!

II. So the question is: How to "solve" a recurrence relation.

A. **Def:**

A linear, homogeneous, recurrence relation of degree 2
with constant coefficients
is$_{def}$
a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2}$$

where $c_1, c_2 \in \mathbf{R}$ & $c_2 \neq 0$

1. See text for complete def of linear homogeneous recurrence relation of degree **k** with constant coefficients.
2. "linear": no exponents
3. "homogeneous": all terms are multiples of the a_i
    - Note: pronounced "homo**JEEN**ee-us", *not* "ho**MOJ**enus", with 5 syllables

4. "constant coefficients":
    they are not functions of n;
    they are constants

B. This determines a *family* of sequences that differ only in their initial conditions.

- E.g.) The Fibonacci recurrence relation:

  $$f_n = f_{n-1} + f_{n-2}$$

  is a linear homegeneous recurrence relation of degree 2
  and can have differing intial conditions,
  yielding different Fibonacci sequences:

  1. $f_0 = 0$ & $f_1 = 1$ yields:

     0,1,1,2,3,5,...

  2. $f_0 = 1$ & $f_1 = 1$ yields:

     1,1,2,3,5,...

  3. $f_0 = 1$ & $f_1 = 2$ yields:

     1,2,3,5,...

C. They are solvable!

III. How do you solve them?

A. The trick:

- Given $a_0=C_0$, $a_1=C_1$, & $a_n=c_1a_{n-1}+c_2a_{n-2}$
  look for solutions of the form $a_n = r^n$, for constant r

B. Why?

1. Because, in the simplest case, when $a_n=c_1a_{n-1}$,
   the *ratio* of sucessive terms is constant;
   ∴ it's a geometric sequence

2. Given the sequence
   $a_0 = C_0$
   $a_n = c_1a_{n-1}$

we have:

$a_0 = C_0$
$a_1 = c_1 C_0$
$a_2 = c_1^2 C_0$
…
$a_n = c_1^n C_0$
$\therefore\ a_n = a_0 c_1^n$

C. $a_n = r^n$ is a solution
(i.e., an "explicit", non-recursive formula)
for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

$\leftrightarrow$

$(a_n =)\ r^n = c_1 r^{n-1} + c_2 r^{n-2}$ (by substituting r^n for a_n)

$\leftrightarrow$

$$\frac{r^n}{r^{n-2}} = \frac{c_1 r^{n-1} + c_2 r^{n-2}}{r^{n-2}} \text{ (for } r \neq 0\text{)}$$

$\leftrightarrow$

$r^2 = c_1 r + c_2$

$\leftrightarrow$

$r^2 - c_1 r - c_2 = 0$ [the **characteristic equation** of the recurrence relation]

$\leftrightarrow$

r is a solution of this equation [the **characteristic root**]
(i.e., makes the equation come out T)

IV. **Thm 1** (p. 462):

- In a theorem, you have to say where everything comes from;
  i.e., you must give their data types.

Let $C_0, C_1 \in \mathbf{N}$ be constants.

Let $a_0 = C_0$ and $a_1 = C_1$ be the initial conditions of a recurrence relation.

Let $c_1, c_2 \in \mathbf{R}$ be such that $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is the recurrence relation.

Let $r_1 \neq r_2$ be 2 distinct roots of the "characteristic equation"

$r^2 - c_1 r - c_2$

of the recurrence relation. Then:

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$(\exists \alpha_1, \alpha_2 \in \mathbf{R})(\forall n \in \mathbf{N})[a_n = \alpha_1 r_1^n + \alpha_2 r_2^n]$

- i.e.) the recurrence relation for the nth term
  can be computed non-recursively

using the formula in terms of α_i and r_i

- This is a non-constructive existence claim!
  - We need an *algorithm* to show how to find the α_i

V. (Outline of) **procedure** (i.e., algorithm)
for solving a (linear homogeneous) **recurrence relation**
(of degree 2 with constant coefficients):

- Details will be given in the next lecture.

- I/P: recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

1. Set up the characteristic equation:

   $r^2 - c_1 r - c_2$

2. Solve the characteristic eqn for r_1, r_2

   a. **if** $r_1 = r_2$,
      then begin O/P "no solution"; halt **end**
      else goto (2b)

   b. Find α_1, α_2 such that $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$:

      a. Use initial conditions to produce 2 simultaneous eqns in 2 unknowns:

      b. Solve these for α_1 & α_2

- O/P: explicit formula for a_n, namely:

  $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$

Next lecture…

[Discrete Structures](#)

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§§7.1–7.2: Recurrence Relations (cont'd)

I. **Thm 1** (p. 462):

Let $C_0, C_1 \in \mathbf{N}$ be constants.

Let $a_0 = C_0$ and $a_1 = C_1$ be the initial conditions for a recurrence relation;
i.e., the first 2 terms of a sequence.

Let $c_1, c_2 \in \mathbf{R}$ be such that $a_n = c_1a_{n-1} + c_2a_{n-2}$ is the recurrence relation.

Let $r_1 \neq r_2$ be 2 distinct roots of the "characteristic equation"

$$r^2 - c_1r - c_2$$

of the recurrence relation.

Then:

$$(\exists\alpha_1, \alpha_2 \in \mathbf{R})(\forall n \in \mathbf{N})[a_n = \alpha_1{r_1}^n + \alpha_2{r_2}^n]$$

- i.e.) the sequence defined *recursively*
  in terms of initial conditions and a recurrence relation
  can be computed *non*-recursively
  using the formula in terms of α_i and r_i

- This is a non-constructive existence claim!
  - We need an *algorithm* to show how to find the α_i

II. **Procedure** (i.e., algorithm) **for solving a** (linear homogeneous) **recurrence relation** (of degree 2 with constant coefficients):

- I/P: recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

1. Set up the characteristic equation:

$$r^2 - c_1 r - c_2$$

2. Solve the characteristic eqn for r_1, r_2

   a. **if** $r_1 = r_2$,
      then begin O/P "no solution"; halt **end**
      else goto (2b)

   b. Use:

$$r_1 = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}$$

and

$$r_2 = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

where $ar^2 + br + c = 0$

i.e.)

$a = 1$
$b = -c_1$
$c = -c_2$

$\therefore$

$$r_1 = \frac{-c_1 + \sqrt{(c_1{}^2 - 4c_2)}}{2}$$

and

$$r_2 = \frac{-c_1 - \sqrt{(c_1{}^2 - 4c_2)}}{2}$$

3. Find α_1, α_2 such that $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$:

   - r_1 & r_2 were computed in step 2

   a. Use initial conditions to produce 2 simultaneous eqns in 2 unknowns:

      i. $a_0 = \alpha_1 r_1{}^0 + \alpha_2 r_2{}^0 = \alpha_1 + \alpha_2$ (!)

      ii. $a_1 = \alpha_1 r_1{}^1 + \alpha_2 r_2{}^1 = \alpha_1 r_1 + \alpha_2 r_2$

b. Solve these for α_1 & α_2

- Note: $\alpha_1 = a_0 - \alpha_2$ (from 3(a)(i))

  $\therefore a_1 = (a_0 - \alpha_2)r_1 + \alpha_2 r_2$ (from 3(a)(ii))
  $= a_0 r_1 - \alpha_2 r_1 + \alpha_2 r_2$
  $= a_0 r_1 + \alpha_2(r_2 - r_1)$

  $\therefore \alpha_2(r_2 - r_1) = a_1 - a_0 r_1$

  $\therefore \alpha_2 = (a_1 - a_0 r_1) / (r_2 - r_1)$

  - **...which is why $r_1 \neq r_2$!**

  $\therefore \alpha_1 = a_0 - \alpha_2 = a_0 - (a_1 - a_0 r_1) / (r_2 - r_1)$
  $= (a_0 r_2 - a_0 r_1 + a_0 r_1 - a_1) / (r_2 - r_1)$
  $= (a_0 r_2 - a_1) / (r_2 - r_1)$

- O/P: explicit formula for a_n, namely:

  $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

III. As an example, let's solve the Fibonacci recurrence relation:

A. $f_0 = 0$
$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

0,1,1,2,3,5,...

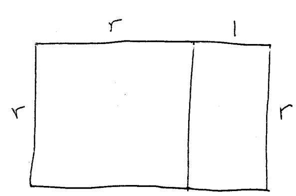
B. Solution:

1. Char Eqn: $r^2 - c_1 r - c_2 = 0$

   - What are c_1, c_2 s.t. $f_n = c_1 f_{n-1} + c_2 f_{n-2}$?
   - Answer: $c_1 = c_2 = 1$
   - $\therefore$ char eqn is: $r^2 - r - 1 = 0$

   Digression:

   - Consider a "golden rectangle":

     —allegedly the most aesthetically pleasing rectangle:

     - The front of the Parthenon has the shape of a golden rectangle.
     - Index cards come in golden rectangles: 3×5; 5×8 (note the Fibonacci numbers!)

- The "golden ratio" (see the picture of the golden rectangle, above) is:

  $$\frac{r+1}{r} = \frac{r}{1}$$

  $\therefore r^2 = r + 1$
  $\therefore r^2 - r - 1 = 0$ [does that look familiar?]

2. Solve for r:

  - $r = (1 \pm \sqrt{(1 + 4)})/2 = (1 \pm \sqrt{5})/2$
  - This is (these are?) the golden ratio(s):
    - $\sqrt{5} = 2.236067977\ldots$
    - $(1+\sqrt{5})/2 = \ \ 1.6180339885\ldots$
    - $(1-\sqrt{5})/2 = -0.6180339885\ldots$
    - Also: the reciprocal of $(1+\sqrt{5})/2 = 0.6180339885\ldots$
      - Weird!

3. Find α_1, α_2 s.t. $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

  - Let $r_1 = (1+\sqrt{5})/2$ & $r_2 = (1-\sqrt{5})/2$
  - $f_0 = 0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 = \alpha_1 + \alpha_2$
  - $f_1 = 1 = \alpha_1 r_1^1 + \alpha_2 r_2^1$

    $= \alpha_1((1+\sqrt{5})/2) + \alpha_2((1-\sqrt{5})/2)$
  - Solve:
    - $0 = \alpha_1 + \alpha_2$
    - $\therefore \alpha_1 = -\alpha_2$
    - $1 = \alpha_1((1+\sqrt{5})/2) + \alpha_2((1-\sqrt{5})/2)$
    - $\therefore 1 = -\alpha_2((1+\sqrt{5})/2) + \alpha_2((1-\sqrt{5})/2)$

      $= \alpha_2((-1-\sqrt{5})/2 + (1-\sqrt{5})/2)$

      $= \alpha_2((-1-\sqrt{5}+1-\sqrt{5})/2)$

      $= \alpha_2((-2\sqrt{5})/2)$

      $= -\alpha_2\sqrt{5}$
    - $\therefore \alpha_2 = -1/\sqrt{5}$
    - $\therefore \alpha_1 = \ \ 1/\sqrt{5}$

4. $\therefore f_n = (1/\sqrt{5})((1+\sqrt{5})/2)^n - (1/\sqrt{5})((1-\sqrt{5})/2)^n$

  - See lecture notes for 11/15/2010, §VI !

IV. Another example:

A. A long time ago, I discovered that if you fold a piece of paper in ½, then fold the top half back, then fold it ¼ of the way, then fold it back and fold it 3/8 of the way, and so on (see Fig. 1, below), the edge of the paper winds up 1/3 of the way from the edge!

1. More precisely, fold it according to the following directions:

1. **Fold *B* to the *left* to coincide with *A* (see fig. 1b); call the crease that is created by this fold, *C* (see fig. 1c).**

2. **Without unfolding the paper, fold *B* to the *right* to coincide with *C*, forming a second crease, *D* (see fig. 1d). Just as a check, the paper (when viewed from the side) should now look like fig. 1e).**

3. **Without unfolding, fold *B* to the *left* to coincide with *D*, creating crease *E* (top view in fig. 1f, side view in fig. 1g).**

4. **Continue as before, successively folding *B* to the right, left, and so on, so that at the end of each folding operation, *B* coincides with the crease made last.**

2.

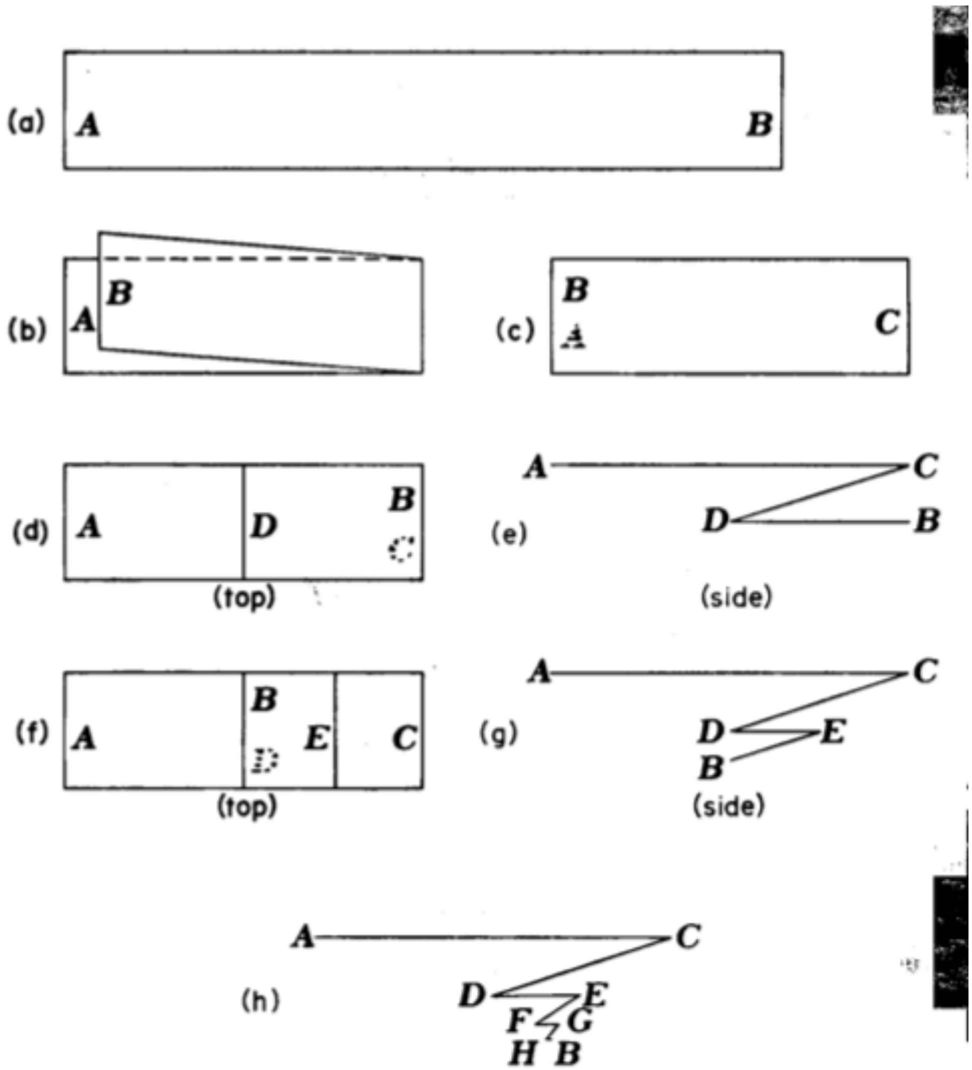


Fig. 1

3. The question is: Where does B end up?

   And the answer is: 1/3 of the way from the edge!

   The puzzle is: How does a sequence of foldings-in-half yield the number 1/3?

B. For the use of *continuous* math (i.e., limits) to prove this, see:

- "Paper Folding and Convergent Sequences" (Rapaport 1974)

C. Here's the sequence of folds, expressed recursively:

$a_0 = 0$ (paper begins unfolded, at one edge, or 0")
$a_1 = ½$ (first fold moves the edge ½, to middle of paper)

$a_n = (a_{n-1} + a_{n-2})/2$

(each subsequent fold moves the edge to the average of its previous two positions)

$= ½a_{n-1} + ½a_{n-2}$

- 0, ½, ¼, 3/8, 5/16, 11/32, …

- The curious feature is that each term in the sequence has a denominator that is a power of 2, yet the limit of the folds is 1/3.

  Where did the "3" come from?

D. Let's solve this recurrence relation:

- I/P: $a_n = ½a_{n-1} + ½a_{n-2}$

1. Char Eqn: $r^2 - ½r - ½$

2. Solve for r:

   - $r = (½ \pm \sqrt{(¼ + 4*½)})/2$
     $= (½ \pm \sqrt{(¼ + 2)})/2$
     $= (½ \pm \sqrt{(9/4)})/2$
     $= (½ \pm 3/2)/2$
     $\in \{1, -½\}$

   - i.e.) $r_1 = 1$ & $r_2 = -½$

3. Solve for α:

   - $a_0 = 0 = \alpha_1 + \alpha_2$

     $\therefore\ \alpha_2 = -\alpha_1$

   - $a_1 = ½ = 1*\alpha_1 + (-½*\alpha_2) = \alpha_1 - \alpha_2/2$

     $\therefore\ ½ = \alpha_1 + \alpha_1/2 = 3\alpha_1/2$

     $\therefore\ 1 = 3\alpha_1$

     $\therefore\ \alpha_1 = 1/3$ [there's the 1/3!!!]

     $\therefore\ \alpha_2 = -1/3$

- O/P:

  $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$
  $= (1/3)*1^n + (-1/3)*(-½)^n =$ $\mathbf{1/3 - (1/3)*(-½)^n}$ $= (1/3)(1 - (-½)^n)$

- (As n gets bigger, the $-½^n$ gets smaller and smaller;
  in the limit it goes to 0, so the limit of the sequence is 1/3)

V.

A. A final example, to be continued next time…

Consider this sequence:

$a_0 = C_0$
$a_1 = C_1$

$$a_n = 3a_{n-1} - 2a_{n-2}, \forall\, n \geq 2$$

Try it! Use the algorithm to show that $a_n = 2^n - 1$, as suggested in the table in the previous lecture.

(answer will be given in the next lecture)

Next lecture…

Lecture Notes, 29 Nov 2010

Last Update: 30 November 2010

Note: NEW or UPDATED material is highlighted

Note: A username and password may be required to access certain documents. Please contact Bill Rapaport.

Index to all lecture notes
…Previous lecture

§§7.1–7.2: Recurrence Relations (cont'd)

I. Answer to the example from last lecture:

A. UPDATED
Consider this sequence:

$a_0 = C_0 = 0$
$a_1 = C_1 = 1$

$a_n = 3a_{n-1} - 2a_{n-2}, \forall\ n \geq 2$

B. Execution of our algorithm:

- I/P: $a_n = 3a_{n-1} - 2a_{n-2}$

1. Char Eqn: $r^2 - 3r + 2 = 0$
2. Solve for r:
   - $r_1 = (3 + \sqrt{}(9 - 4*2))/2 = (3+\sqrt{}1)/2 = 2$
   - $r_2 = (3 - \sqrt{}(9 - 4*2))/2 = (3-\sqrt{}1)/2 = 1$
3. Find α_i:
   - $a_0 = \alpha_1*2^0 + \alpha_2*1^0 = \alpha_1 + \alpha_2 = 0$

     $\therefore\ \alpha_1 = -\alpha_2$
   - $a_1 = \alpha_1*2^1 + \alpha_2*1^1 = 2\alpha_1 + \alpha_2 = 1$

     $\therefore\ 2(-\alpha_2) + \alpha_2 = 1$

$\therefore -\alpha_2 = 1$

$\therefore \alpha_2 = -1$

$\therefore \alpha_1 = 1$

- O/P:

$$\therefore a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$= 1*2^n + (-1)*1^n$$
$$= 2^n - 1$$

C. You should try this for the other initial conditions shown in the table in the previous lecture to make sure that you get the same answers that I did.

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§8.1 and §8.5: Relations

I. In language, including languages from math & CS (such as programming languages), there are:

A. noun phrases:

- individual terms & descriptions, including proper names
  - e.g.) 'Fido', 'Prof. Rapaport', '2'
- general terms and descriptions
  - e.g.) 'dog', 'professor', 'real number'

B. verb phrases:

- e.g.) 'sees', 'runs', 'divides', '<'

C. adjective phrases & prepositional phrases:

- e.g.) 'is red', 'is even'

1. We can model these mathematically as follows:

   a. objects in the domain: model individual NPs
   b. sets of objects: model general NPs, intransitive VPs, adjective phrases, properties
   c. n-ary relations (Cartesian products of sets): model transitive VPs, relational properties, prepositions

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II.

A. Def:

Let A,B be sets.
Then R **is a binary relation on** A **and** B
=def
$R \subseteq A \times B$

B. Notation:

Let $a \in A$, $b \in B$.