This document consists of 3 parts:

1. Rapaport, William J. (1978), "An Algebraic Interpretation of Deontic Logic; Part I: Sentential Fragment" (unpublished ms.)

2. In the interests of honesty, appended is the negative referee report and rejection letter from the Journal of Symbolic Logic, pointing out some errors.

3. A note from a former colleague, Jerzy Pogonowski, then of the Dept. of Lingustics, SUNY Buffalo, and (then and now), Department of Applied Logic, Adam Mickiewicz University, Poznan, Poland (http://www.logic.amu.edu.pl/history.html), suggesting some corrections.

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I. PHILOSOPHICAL PRELIMINARIES

1. Introduction

In this essay, we investigate generalizations of two well-known algebraic structures, the vector space and the module, which arise in logics associated with certain fragments of natural language. While we shall consider certain _logical_ problems from an algebraic viewpoint, we will also be interested in what our results say about the mathematical structure of _natural language_--in particular, the structure of ethical language.

2. Castañeda's Deontic Logic

The crucial feature of ethical language to be examined here is most easily illustrated by considering commands. When the indicative sentence 'It is raining' and the imperative sentence 'Go home!' are combined in a conditional construction, the resulting sentence, 'If it is raining, go home!', appears to be an imperative. (The same phenomenon occurs in erotetic logic--the logic of questions. We shall return to this later.)

While this feature of language has been noted implicitly by some authors (e.g., Rescher 1966, Ch. 3; Harrah 1961: 44), its most explicit statement appears in the writings of

Hector-Neri Castañeda (1974, 1975). Moreover, the analysis of deontic language (the language of ethics) presented by Castañeda makes an attempt to explain this feature in terms of metaphysical theses about language and predication. For these reasons, we shall take his system as the basis for our algebraic interpretation.

It will prove useful to present briefly (though we shall not defend) some of the notions of Castañeda's deontic logic; a more formal presentation appears in Section II.

Many philosophers hold that the units of thinking (and hence of language) are "propositions" and that, variously, there is either no difference or else little or no connection between purely contemplative (or theoretical) thinking and practical thinking (the sort that results in an action by the thinker).[2] Castañeda, on the other hand, believes that, while propositions are indeed the units of contemplative thinking, what he calls "practitions" are the units of practical thinking and that there is a fundamental unity (though not an identity) between these kinds of thinking.

More precisely, a <u>proposition</u> is a truth-valued unit of contemplative thinking, and propositions of the simplest kind are expressed by declarative sentences consisting of a noun phrase (NP) plus a verb phrase (VP) in the indicative mood (Castañeda 1974: 29ff, 109; 1975: 6, 44, 260f).

A <u>practition</u> is the fundamental unit of practical thinking and comes in two varieties: (1) A <u>prescription</u> is the "common structure at the core" of orders, requests, pieces of advice, entreaties, etc., "that demand of the same agents the same actions in the same circumstances." They are not propositions, and a prescription of the simplest kind is expressed (according to Castañeda) by a NP in the second- or third-person plus an infinitive VP, e.g., 'John to go home'. (2) An intention is the first-person counterpart of a prescription. It, too, is not a proposition and is typically expressed by 'I' plus an infinitive VP. These NP + VP constructions are called practitives. (Castañeda 1974: 40ff, 93f, 109; 1975: 7, 43f, 91ff, 154, 172, 260f.)

The crucial difference between propositions and practitions is in their respective modes of predication. Presumably, the

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propositional copula unites (almost) any subject with (almost) any predicate; the practitional copula unites agents with actions and, moreover, is <u>different</u> from the propositional copula. According to Castañeda, it is "a complex of the ordinary copula and a practitional operator" or "a signal of its practical or causal openness." The ordinary, or propositional, copula "combines the subject and the predicate in a way suitable for possession of truth," while the practitional copula "combines its subject and its predicate in a way suitable for action." (Castañeda 1974: 76, 93f; 1975: 96ff, 170ff, 280ff.)[3]

The unit of thinking, be it practical or contemplative, is called a <u>noema</u>. Thus, both propositions and practitions are noemata(Castañeda 1974: 29; 1975: 7).

Finally, a <u>deontic judgment</u> is a proposition formed by applying deontic operators (such as <u>ought</u> or <u>it is permissible that</u>) to a practition. They are, thus, practical noemata(because suitable "for the guidance of conduct") of such forms as 'agent <u>X</u> ought to do <u>A</u>' or 'it is wrong that <u>X</u> do <u>A</u>'. (Castañeda 1974: 42, 97; 1975: 178f, 204, 261.)[4]

The fundamental principle with which we are presently concerned is:

(P4) Compounds made up of both propositions and practitions not in the scope of deontic properties [i.e., operators] are practitions. (Castañeda 1974: 79; cf. 1974: 37, 80 and 1975: 174, 256.)

Several explanations or partial explanations are offered for the truth of (P4). First, Castañeda 1974: 29 can be read as suggesting that (P4) is a reflection of the fact that practical thinking "includes" contemplative thinking; but it is not immediately clear how the connection with (P4) might be made.

Second, (P4) might be a consequence of the existence of two modes of predication, though it would need to be argued that the combination of a proposition and a practition yields something which is not capable of having a truth value but <u>is</u> capable of guiding action (cf. Castañeda 1975: 280).

Third, there is Castañeda's explicit explanation:

> [T]he mind is presented with propositions. But sometimes the mind is so set that it "marks" or "spots" . . . certain components of those propositions; these "spotted" propositions are practitions, and the mind apprehends them. (Castañeda 1975: 283.)

Thus, if one component of a mixed compound is "spotted", so is the compound. But the nature of "spotting" requires elaboration before this explanation can be complete.

The fourth and, in my opinion, best explanation begins by considering the abstract possibilities for the nature of mixed compounds: they could be practitions (as in (P4)), propositions, or some third thing. But humans are uninterested in mixed compounds of the latter two types, presumably because humans "are primarily agents in, and only derivatively contemplators of, the universe, [and] are fundamentally concerned with actions" (Castañeda 1975: 112f).[5] The defense of <u>this</u> is, as Castañeda recognizes, another story; but an explanation in this direction seems to me to offer the best account within his metaphysical framework of all the phenomena involved in the other explanations.

My goal in the present essay is to provide an algebraic interpretation--which, it is hoped, will be of some independent interest--of the sentential fragment of Castañeda's deontic logic.[6] It is expected that the extension of these results to the first-order and modal fragments (i.e., to quantifiers and to deontic operators such as the ought-to-do[7]) will be relatively straightforward once the groundwork is laid out here.

Once again, let me stress two points. First, the foregoing is not intended as a defense of Castañeda's theory. Second, I believe that his theory is the clearest (though by no means the only) one available for the algebraic-logical investigations that follow.

II. SYNTAX OF SYSTEM C

Castañeda considers a family of deontic logics, one for each ought-to-do operator. On the sentential level, there are only two connectives, negation and conjunction, but disjunction and material implication can be defined in his system in the usual way. As axioms, he takes all truth-table tautologies. (Castañeda 1974: 109ff; 1975: 260ff.)

For our purposes, we need not be concerned with more than one member of this family of logics, so the system C to be presented here can be considered as a sentential fragment of (any) one, representative member. It will also prove convenient for the algebraic interpretation to take negation, conjunction, disjunction, and material implication as primitive connectives of C and to use a somewhat less generous axiom set.

The alphabet of the underlying formal language of C consists of:

(1) A denumerable set, $\underline{V}_{\underline{I}}$, of indicative (i.e., "propositional") variables with numerical subscripts:

$\varphi_1, \varphi_2, \ldots$.

(2) A denumerable set, $\underline{V}_{\underline{P}}$, of pure (i.e., unmixed) practitive variables, with numerical subscripts, which is disjoint from $\underline{V}_{\underline{I}}$:

$\underline{X}_1, \underline{X}_2, \ldots$.

(3) Three disjoint sets of connectives: A set of internal, indicative connectives, $\{\neg_{\underline{i}}, \wedge_{\underline{i}}, \vee_{\underline{i}}, \rightarrow_{\underline{i}}\}$; a set of internal, practitive connectives, $\{\neg_{\underline{p}}, \wedge_{\underline{p}}, \vee_{\underline{p}}, \rightarrow_{\underline{p}}\}$; and a set of external connectives, $\{\wedge_{\underline{e}}, \vee_{\underline{e}}, \rightarrow_{\underline{e}}\}$.

(4) Parentheses:[8] (,).

While Castañeda favors conflating the connectives on the grounds that their logical relationships are analogous, the algebraic interpretation will be clearer if we retain the distinctions. (In his terminology, we are doing "sh-logic"; Castañeda 1974: 88; 1975: 101f.) Moreover, it is conceivable that the "deep structure" of our ethical language has these distinctions, and they are conflated only in the "surface structure".

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The <u>formation rules</u> are as follows:

(a) The set <u>WFI</u> of <u>well-formed indicatives</u> (wfis) is the smallest set such that

(i) $\underline{V}_{\underline{I}} \subset \underline{WFI}$, and

(ii) if $\underline{p}$, $\underline{q} \in \underline{WFI}$, then $(\neg_{\underline{i}}\underline{p})$, $(\underline{p} \wedge_{\underline{i}} \underline{q})$, $(\underline{p} \vee_{\underline{i}} \underline{q})$, $(\underline{p} \rightarrow_{\underline{i}} \underline{q}) \in \underline{WFI}$.

(b) The set <u>WFP</u> of <u>well-formed practitives</u> (wfps) is the smallest set such that

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(i) $\underline{V_p} \in \underline{WFP}$, and

(ii) if $\underline{A}, \underline{B} \in \underline{WFP}$, then $(\neg_{\underline{p}}\underline{A})$, $(\underline{A} \wedge_{\underline{p}} \underline{B})$, $(\underline{A} \vee_{\underline{p}} \underline{B})$, $(\underline{A} \rightarrow_{\underline{p}} \underline{B}) \in \underline{WFP}$, and

(iii) if $\underline{p} \in \underline{WFI}$ and $\underline{A} \in \underline{WFP}$, then $(\underline{p} \wedge_{\underline{e}} A)$, $(\underline{A} \wedge_{\underline{e}} \underline{p})$, $(\underline{p} \vee_{\underline{e}} \underline{A})$, $(\underline{A} \vee_{\underline{e}} \underline{p})$, $(\underline{p} \rightarrow_{\underline{e}} \underline{A})$, $(\underline{A} \rightarrow_{\underline{e}} \underline{p}) \in \underline{WFP}$.

(We let $\underline{p}$, $\underline{q}$, ... and $\underline{A}$, $\underline{B}$, ... be schematic letters ranging over wfis and wfps, respectively.)

(c) The set $\underline{WFN}$ of **well-formed noemata** (wfns) $=_{\underline{df}}$ $\underline{WFI} \cup \underline{WFP}$.

Note that $\underline{WFI}$ and $\underline{WFP}$ are disjoint. (And we will omit parentheses when no ambiguity results.)

To present the axiom set compactly, we introduce the following notation: lower-case Greek letters $\alpha, \beta, \gamma, \ldots$ are variables ranging over schematic letters which, in turn, range over wfns, and non-subscripted connectives are variables ranging over subscripted connectives. Thus, axiom schema A2, below, is really a "super" schema among whose "subschemata" are:

$\underline{p} \rightarrow_{\underline{i}} (\underline{q} \vee_{\underline{i}} \underline{p})$

$\underline{p} \rightarrow_{\underline{e}} (\underline{A} \vee_{\underline{e}} \underline{p})$

$\underline{A} \rightarrow_{\underline{e}} (\underline{q} \vee_{\underline{e}} \underline{A})$

$\underline{A} \rightarrow_{\underline{p}} (\underline{B} \vee_{\underline{p}} \underline{A})$,

etc. The axiom schemata of $\underset{\sim}{C}$ (cf. Rasiowa and Sikorski 1963: 168f) are:

(A1) $(\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)]$

(A2) $\alpha \rightarrow (\alpha \vee \beta)$

(A3) $\beta \rightarrow (\alpha \vee \beta)$

(A4) $(\alpha \rightarrow \gamma) \rightarrow [(\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)]$

(A5) $(\alpha \wedge \beta) \rightarrow \alpha$

(A6) $(\alpha \wedge \beta) \rightarrow \beta$

(A7) $(\gamma \rightarrow \alpha) \rightarrow [(\gamma \rightarrow \beta) \rightarrow (\gamma \rightarrow (\alpha \wedge \beta))]$

(A8) $[\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \wedge \beta) \rightarrow \gamma]$

(A9) $[(\alpha \wedge \beta) \rightarrow \gamma] \rightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)]$

(A10) $(\alpha \wedge \neg \alpha) \rightarrow \beta$

(A11) $[\alpha \rightarrow (\alpha \wedge \neg \alpha)] \rightarrow \neg \alpha$

(A12) $\alpha \vee \neg \alpha$

There is a "super" schematic rule of inference (MP) whose four subschemata are:

$$\begin{array}{l} \underline{p} \\ \underline{p} \rightarrow_{\underline{i}} \underline{q} \\ \hline \underline{q} \end{array} \qquad \begin{array}{l} \underline{p} \\ \underline{p} \rightarrow_{\underline{e}} \underline{A} \\ \hline \underline{A} \end{array} \qquad \begin{array}{l} \underline{A} \\ \underline{A} \rightarrow_{\underline{e}} \underline{q} \\ \hline \underline{q} \end{array} \qquad \begin{array}{l} \underline{A} \\ \underline{A} \rightarrow_{\underline{p}} \underline{B} \\ \hline \underline{B} \end{array}$$

III. ALGEBRAIC INTERPRETATION OF $\underset{\sim}{C}$

1. Dominance Algebras

Principle (P4) is analogous to scalar multiplication in vector spaces: the scalar product of a scalar and a vector is always a vector. Thus, propositions play a scalar role, practitions play a vector role. But neither a vector space nor its standard generalization, a module, will do for the algebraic interpretation of $\underset{\sim}{C}$, since vector spaces and modules have only four operations where we will need eleven. Rather, we shall be interested in special cases of what will be called, for reasons to be elaborated upon in Section IV, a "dominance algebra", viz., a system "consisting" of two algebras, one of which is "dominant" over the other (the "recessive" algebra) in the sense that the results of "externally" combining members of the recessive algebra with members of the dominant algebra will be members of the dominant algebra. (Vectors, e.g., are dominant over scalars.)

DEF 1: $\langle \underline{M}, \underline{R}, \underline{I}, \underline{E} \rangle$ is a dominance algebra (over $\underline{R}$) iff

(i) $\underline{R}$ is an abstract algebra

(ii) $\underline{M} \neq \emptyset$

(iii) $\underline{I}$ is a non-empty set of "internal" operations such that for all $\underline{i} \in \underline{I}$, $\underline{i}:\underline{M}^{\underline{n}} \rightarrow \underline{M}$ (for $\underline{n} \in \omega$)

(iv) $\underline{E}$ is a non-empty set of "external" operations such that for all $\underline{e} \in \underline{E}$, $\underline{e}:(\underline{R}^{\underline{n}} \times \underline{M}^{\underline{m}}) \cup (\underline{M}^{\underline{m}} \times \underline{R}^{\underline{n}}) \rightarrow \underline{M}$ (for $\underline{m}, \underline{n} \in \omega$).

Modules (and, hence, vector spaces) are dominance algebras, for $\langle \underline{M}, \underline{R}, \{+\}, \{\cdot\} \rangle$ is a module over $\underline{R}$ iff (i) $\underline{R}$ is a ring (hence, an abstract algebra), (ii) $M \neq \emptyset$, (iii) $+: M^2 \to M$ is an internal operation such that $\langle M, + \rangle$ is an abelian group, and (iv) $.: (R \times M) \to M$ is an external operation such that $r(a+b) = ra + rb$, $r(sa) = (rs)a$, and $(r+s)a = ra + sa$, for all $a, b \in M$ and $r, s \in R$ (here and henceforth suppressing the '.' in favor of juxtaposition, for legibility). (Cf. Herstein 1964: 160, and, especially, Solian 1977: 30f; cf. also Sect. IV below.)

The first special case of interest is that of a "Boolean" dominance algebra:

DEF 2: $\langle M, B, I, E \rangle$ is a Boolean dominance algebra (over B) iff it is a dominance algebra (over B) and B is a Boolean algebra.

For convenience, we will use the following liberal characterization of Boolean algebras (cf. Rasiowa and Sikorski 1963: 71):

DEF 3: $\langle B, \{+, ., \Rightarrow, -\} \rangle$ is a Boolean algebra iff

(i) $B \neq \emptyset$

(ii) $+: B^2 \to B$, $.: B^2 \to B$, $\Rightarrow: B^2 \to B$, and $-: B \to B$ are (internal) operations such that, for all $a, b, c \in B$,

(a)	$a+b = b+a$	(a')	$ab = ba$
(b)	$a+(b+c) = (a+b)+c$	(b')	$a(bc) = (ab)c$
(c)	$ab+b = b$	(c')	$a(a+b) = a$
(d)	$a(b+c) = ab+ac$	(d')	$a+bc = (a+b)(a+c)$
(e)	$a(-a)+b = b$	(e')	$(a+(-a))b = b$

(f) $a \Rightarrow b = -a+b$

In the second special case of interest, both constitutive algebras are Boolean:

<u>DEF 4</u>: Let $\underline{I} = \{\#, \times, \supset, \sim\}$.

Let $\underline{E} = \{\pm, \underline{\times}, \underline{\Rightarrow}\}$.

Then $\langle \underline{M}, \underline{B}, \underline{I}, \underline{E} \rangle$ <u>is a double Boolean dominance algebra</u> (<u>over</u> $\underline{B}$) iff

(i) $\langle \underline{B}, \{+, \cdot, \Rightarrow, -\} \rangle$ is a Boolean algebra

(ii) $\langle \underline{M}, \underline{I} \rangle$ is a Boolean algebra

(iii) $\pm, \underline{\times}, \underline{\Rightarrow} : (\underline{M} \times \underline{B}) \cup (\underline{B} \times \underline{M}) \to \underline{M}$ are external operations such that, for all $p, q \in \underline{B}$ and $A, B \in \underline{M}$,

(a) $p \pm A = A \pm p$

(a′) $p \,\underline{\times}\, A = A \,\underline{\times}\, p$

(b)
$$p \pm (q \pm A) = (p+q) \pm A$$
$$p \pm (A \pm q) = (p \pm A) \pm q$$
$$A \# (p \pm B) = (A \pm p) \# B$$
$$A \# (B \pm p) = (A \# B) \pm p$$
$$p \pm (A \# B) = (p \pm A) \# B$$
$$A \pm (p+q) = (A \pm p) \pm q$$

(b′)
$$p \,\underline{\times}\, (q \,\underline{\times}\, A) = (pq) \,\underline{\times}\, A$$
$$p \,\underline{\times}\, (A \,\underline{\times}\, q) = (p \,\underline{\times}\, A) \,\underline{\times}\, q$$
$$A \times (p \,\underline{\times}\, B) = (A \,\underline{\times}\, p) \times B$$
$$A \times (B \,\underline{\times}\, p) = (A \times B) \,\underline{\times}\, p$$
$$p \,\underline{\times}\, (A \times B) = (p \,\underline{\times}\, A) \times B$$
$$A \,\underline{\times}\, (pq) = (A \,\underline{\times}\, p) \,\underline{\times}\, q$$

(c) $(p \,\underline{\times}\, A) \# A = A$

(c′) $A \,\underline{\times}\, (A \pm p) = A$

(d)
$$p \,\underline{\times}\, (q \pm A) = (pq) \pm (p \,\underline{\times}\, A)$$
$$p \,\underline{\times}\, (A \pm q) = (p \,\underline{\times}\, A) \pm (pq)$$
$$A \times (p \pm B) = (A \,\underline{\times}\, p) \# (A \times B)$$
$$A \times (B \pm p) = (A \times B) \# (A \,\underline{\times}\, p)$$
$$p \,\underline{\times}\, (A \# B) = (p \,\underline{\times}\, A) \# (p \,\underline{\times}\, B)$$
$$A \,\underline{\times}\, (p+q) = (A \,\underline{\times}\, p) \# (A \,\underline{\times}\, q)$$

(d′)
$$p \pm (q \,\underline{\times}\, A) = (p+q) \,\underline{\times}\, (p \pm A)$$
$$p \pm (A \,\underline{\times}\, q) = (p \pm A) \,\underline{\times}\, (p+q)$$
$$A \# (p \,\underline{\times}\, B) = (A \pm p) \times (A \# B)$$
$$A \# (B \,\underline{\times}\, p) = (A \# B) \times (A \pm p)$$
$$p \pm (A \times B) = (p \pm A) \times (p \pm B)$$
$$A \pm (pq) = (A \pm p) \times (A \pm q)$$

(e) $p(-p) \pm A = A$

(e′) $(p + (-p)) \,\underline{\times}\, A = A$

(f)
$$p \,\underline{\Rightarrow}\, A = -p \pm A$$
$$A \,\underline{\Rightarrow}\, p = \sim A \pm p$$

A couple of remarks are in order. First, four combinatorially possible equations are missing from this list, viz.,

$(\underline{A}\underline{X}\underline{p})\underline{+}\underline{p} = \underline{p}$

$\underline{p}\underline{X}(\underline{p}\underline{+}\underline{A}) = \underline{p}$

$(\underline{A}X(\sim\underline{A}))\underline{+}\underline{p} = \underline{p}$

$(\underline{A}\#(\sim\underline{A}))\underline{X}\underline{p} = \underline{p}$,

because each violates the distinction between "scalars" (members of $\underline{B}$) and "vectors" (members of $\underline{M}$) (cf. n.9).

Second, the items on the second and fourth lines under (d) and (d') are redundant in view of the commutativity of $\underline{+}$ and $\underline{X}$.

2. The Lindenbaum Algebra of Noemata

We can transform our formulation of the noematic logic $\underset{\sim}{C}$ (cf. n.6) into an algebra by taking algebraic operations corresponding to the connectives of $\underset{\sim}{C}$ in a natural way. Thus, the <u>algebra of noemata</u>, <u>AN</u>, is $\langle \underline{WFP}, \underline{WFI}, \{\neg^*_{\underline{i}}, \wedge^*_{\underline{i}}, \vee^*_{\underline{i}}, \rightarrow^*_{\underline{i}}\}, \{\neg^*_{\underline{p}}, \wedge^*_{\underline{p}}, \vee^*_{\underline{p}}, \rightarrow^*_{\underline{p}}\}, \{\wedge^*_{\underline{e}}, \vee^*_{\underline{e}}, \rightarrow^*_{\underline{e}}\}\rangle$, where, e.g., the operation $\neg^*_{\underline{i}}$ is defined so that $\neg^*_{\underline{i}}\underline{p} = \neg_{\underline{i}}\underline{p}$, etc. In what follows, we may and shall omit the * for convenience.

Of course, <u>AN</u> is not a double Boolean dominance algebra (DBDA), since, e.g., $\underline{p} \wedge_{\underline{i}} \underline{p} \neq \underline{p}$ (for all $\underline{p} \in \underline{WFP}$). If it were, our task would be over. However, $(\underline{p} \wedge_{\underline{i}} \underline{p}) \leftrightarrow_{\underline{i}} \underline{p}$ is a theorem of $\underset{\sim}{C}$ (where $\underline{p} \leftrightarrow_{\underline{i}} \underline{q} =_{\underline{df}} (\underline{p} \rightarrow_{\underline{i}} \underline{q}) \wedge_{\underline{i}} (\underline{q} \rightarrow_{\underline{i}} \underline{p})$, and similarly for $\leftrightarrow_{\underline{p}}, \leftrightarrow_{\underline{e}}$).

In order to show the relationship between $\underset{\sim}{C}$ and DBDAs, we need to conflate items like $\underline{p} \wedge_{\underline{i}} \underline{p}$ with $\underline{p}$. This is done by the following technique due to Lindenbaum (cf. Rasiowa and Sikorski 1963: 209, 245). We first introduce equivalence relations on AN as follows:[9]

DEF 5: $p \equiv_i q$ iff $p \leftrightarrow_i q$ is a theorem of C,

$A \equiv_p B$ iff $A \leftrightarrow_p B$ is a theorem of C.

Then, where '$\equiv$' is a schematic relation symbol ranging over $\equiv_i$ and $\equiv_p$; where $\alpha, \beta, \gamma, \delta$ are schematic letters ranging over wfis and wfps; and where $\neg$, $\wedge$, $\vee$, $\rightarrow$ are schematic connectives as before, the following hold (as can be shown via A1-A12 and MP):

(E) (i) $\alpha \equiv \alpha$

(ii) if $\alpha \equiv \beta$, then $\beta \equiv \alpha$

(iii) if $\alpha \equiv \beta$ and $\beta \equiv \gamma$, then $\alpha \equiv \gamma$

(iv) if $\alpha \equiv \beta$, then $\neg\alpha \equiv \neg\beta$

(v) if $\alpha \equiv \beta$ and $\gamma \equiv \delta$, then $(\alpha \wedge \gamma) \equiv (\beta \wedge \delta)$

(vi) if $\alpha \equiv \beta$ and $\gamma \equiv \delta$, then $(\alpha \vee \gamma) \equiv (\beta \vee \delta)$

(vii) if $\alpha \equiv \beta$ and $\gamma \equiv \delta$, then $(\alpha \rightarrow \gamma) \equiv (\beta \rightarrow \delta)$

(viii) $(\alpha \wedge \alpha) \equiv \alpha$ (viii′) $(\alpha \vee \alpha) \equiv \alpha$

(ix) $(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$ (ix′) $(\alpha \vee \beta) \equiv (\beta \vee \alpha)$

(x) $\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma$ (x′) $\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$

(xi) $\alpha \wedge (\alpha \vee \beta) \equiv \alpha$ (xi′) $\alpha \vee (\alpha \wedge \beta) \equiv \alpha$

(xii) $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ (xii′) $\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

(xiii) $(\alpha \wedge \neg\alpha) \vee \beta \equiv \beta$ (xiii′) $(\alpha \vee \neg\alpha) \wedge \beta \equiv \beta$

(xiv) $\alpha \rightarrow \beta \equiv \neg\alpha \vee \beta$

By (i)-(iii), $\equiv_p$ and $\equiv_i$ are equivalence relations.

The Lindenbaum algebra for <u>AN</u>, $\mathcal{L}_{\underline{AN}}$, is simply the quotient algebra $\underline{AN}/_{\equiv}$. To construct it, we need the following definitions of, respectively, equivalence classes of wfis and wfps, quotient sets of wfis and wfps, and operations on and between those quotient sets:

<u>DEF 6</u>: $|p| =_{df} \{q \in \underline{WFI} : p \equiv_i q\}$

$$|A| =_{df} \{B \in \underline{WFP} : A \equiv_p B\}$$

<u>DEF 7</u>: $\underline{WFI}/_{\equiv_i} =_{df} \{|p| : p \in \underline{WFI}\}$

$$\underline{WFP}/_{\equiv_p} =_{df} \{|A| : A \in \underline{WFP}\}$$

<u>DEF 8</u>: $\text{\textcircled{$\neg_i$}}\,|p| =_{df} |\neg_i p|$ $\qquad$ $\text{\textcircled{$\neg_p$}}\,|A| =_{df} |\neg_p A|$

$$|p|\ \text{\textcircled{$\wedge_i$}}\ |q| =_{df} |p \wedge_i q| \qquad |A|\ \text{\textcircled{$\wedge_p$}}\ |B| =_{df} |A \wedge_p B|$$

$$|p|\ \text{\textcircled{$\vee_i$}}\ |q| =_{df} |p \vee_i q| \qquad |A|\ \text{\textcircled{$\vee_p$}}\ |B| =_{df} |A \vee_p B|$$

$$|p|\ \text{\textcircled{$\rightarrow_i$}}\ |q| =_{df} |p \rightarrow_i q| \qquad |A|\ \text{\textcircled{$\rightarrow_p$}}\ |B| =_{df} |A \rightarrow_p B|$$

$$|p|\ \text{\textcircled{$\wedge_e$}}\ |A| =_{df} |p \wedge_e A|$$

$$|A|\ \text{\textcircled{$\wedge_e$}}\ |p| =_{df} |A \wedge_e p|$$

$$|p|\ \text{\textcircled{$\vee_e$}}\ |A| =_{df} |p \vee_e A|$$

$$|A|\ \text{\textcircled{$\vee_e$}}\ |p| =_{df} |A \vee_e p|$$

$$|p|\ \text{\textcircled{$\rightarrow_e$}}\ |A| =_{df} |p \rightarrow_e A|$$

$$|A|\ \text{\textcircled{$\rightarrow_e$}}\ |p| =_{df} |A \rightarrow_e p|$$

Before presenting the definition of $\mathcal{L}_{\underline{AN}}$, we must make sure that these operations are well-defined (i.e., single-valued). Several lemmata, of which the following are representative, can be proved:

LEM 1: If $\underline{p} \equiv_{\underline{i}} \underline{q}$, then $\textcircled{\neg_{\underline{i}}}|\underline{p}| = \textcircled{\neg_{\underline{i}}}|\underline{q}|$.

proof: If $\underline{p} \equiv_{\underline{i}} \underline{q}$, then $\neg_{\underline{i}}\underline{p} \equiv_{\underline{i}} \neg_{\underline{i}}\underline{q}$, by an instance of (Eiv).

Hence, $|\neg_{\underline{i}}\underline{p}| = \{\underline{r} \in \underline{WFI}: \neg_{\underline{i}}\underline{p} \equiv_{\underline{i}} \underline{r}\} = \{\underline{r} \in \underline{WFI}: \neg_{\underline{i}}\underline{q} \equiv_{\underline{i}} \underline{r}\}$

$= |\neg_{\underline{i}}\underline{q}|$, by instances of (Eii) and (Eiii).

But, by the definition of $\textcircled{\neg_{\underline{i}}}$, $\textcircled{\neg_{\underline{i}}}|\underline{p}| = |\neg_{\underline{i}}\underline{p}| = |\neg_{\underline{i}}\underline{q}|$

$= \textcircled{\neg_{\underline{i}}}|\underline{q}|$ ∎

Similar lemmata can be proved for $\textcircled{\wedge_{\underline{i}}}$, $\textcircled{\vee_{\underline{i}}}$, $\textcircled{\rightarrow_{\underline{i}}}$, $\textcircled{\neg_{\underline{p}}}$, $\textcircled{\wedge_{\underline{p}}}$, $\textcircled{\vee_{\underline{p}}}$, and $\textcircled{\rightarrow_{\underline{p}}}$. That $\textcircled{\wedge_{\underline{e}}}$, $\textcircled{\vee_{\underline{e}}}$, and $\textcircled{\rightarrow_{\underline{e}}}$ are well-defined can be proved by lemmata such as this:

LEM 2: If $\underline{p} \equiv_{\underline{i}} \underline{q}$ and $\underline{A} \equiv_{\underline{p}} \underline{B}$, then $|\underline{p}| \textcircled{\wedge_{\underline{e}}} |\underline{A}| = |\underline{q}| \textcircled{\wedge_{\underline{e}}} |\underline{B}|$.

proof: If $\underline{p} \equiv_{\underline{i}} \underline{q}$ and $\underline{A} \equiv_{\underline{p}} \underline{B}$, then $(\underline{p} \wedge_{\underline{e}} \underline{A}) \equiv_{\underline{p}} (\underline{q} \wedge_{\underline{e}} \underline{B})$, by an instance of (Ev). Hence, $|\underline{p} \wedge_{\underline{e}} \underline{A}| = \{\underline{C} \in \underline{WFP}: (\underline{p} \wedge_{\underline{e}} \underline{A}) \equiv_{\underline{p}} \underline{C}\} =$ $= \{\underline{C} \in \underline{WFP}: (\underline{q} \wedge_{\underline{e}} \underline{B}) \equiv_{\underline{p}} \underline{C}\} = |\underline{q} \wedge_{\underline{e}} \underline{B}|$. But, by the definition of $\textcircled{\wedge_{\underline{e}}}$, $|\underline{p}| \textcircled{\wedge_{\underline{e}}} |\underline{A}| = |\underline{p} \wedge_{\underline{e}} \underline{A}| = |\underline{q} \wedge_{\underline{e}} \underline{B}| =$ $= |\underline{q}| \textcircled{\wedge_{\underline{e}}} |\underline{B}|$ ∎

Thus, (Eiv)-(Evii) insure that these operations are well-defined and permit the following definition and theorem:

DEF 9: $\mathcal{L}_{\underline{AN}} =_{\underline{df}} \langle {}^{WFP}/_{\equiv_{\underline{p}}}, {}^{WFI}/_{\equiv_{\underline{i}}}, \{ \textcircled{\vee_{\underline{p}}}, \textcircled{\wedge_{\underline{p}}}, \textcircled{\rightarrow_{\underline{p}}}, \textcircled{\neg_{\underline{p}}} \}, \{ \textcircled{\vee_{\underline{e}}}, \textcircled{\wedge_{\underline{e}}}, \textcircled{\rightarrow_{\underline{e}}} \} \rangle.$

THM 1: $\mathcal{L}_{\underline{AN}}$ is a DBDA.

proof: (i) Show $\langle {}^{WFI}/_{\equiv_{\underline{i}}}, \{ \textcircled{\vee_{\underline{i}}}, \textcircled{\wedge_{\underline{i}}}, \textcircled{\rightarrow_{\underline{i}}}, \textcircled{\neg_{\underline{i}}} \} \rangle$ is a Boolean algebra:

Clearly, ${}^{WFI}/_{\equiv_{\underline{i}}}$ is non-empty. It only needs to be shown that the operations satisfy (a)-(f) and (a')-(e') of the definition of Boolean algebras. Here we will only show, as a representative, that $\textcircled{\wedge_{\underline{i}}}$ satisfies (a'):

Show $|\underline{p}| \textcircled{\wedge_{\underline{i}}} |\underline{q}| = |\underline{q}| \textcircled{\wedge_{\underline{i}}} |\underline{p}|$:

$$|\underline{p}| \textcircled{\wedge_{\underline{i}}} |\underline{q}| = |\underline{p} \wedge_{\underline{i}} \underline{q}|, \text{ by definition}$$
$$= |\underline{q} \wedge_{\underline{i}} \underline{p}|, \text{ by (Eix)}$$
$$= |\underline{q}| \textcircled{\wedge_{\underline{i}}} |\underline{p}|, \text{ by definition.}$$

(ii) That $\langle {}^{WFP}/_{\equiv_{\underline{p}}}, \{ \textcircled{\vee_{\underline{p}}}, \textcircled{\wedge_{\underline{p}}}, \textcircled{\rightarrow_{\underline{p}}}, \textcircled{\neg_{\underline{p}}} \} \rangle$ is a Boolean algebra can be shown in a similar fashion.

(iii) Show $\textcircled{\vee_{\underline{e}}}, \textcircled{\wedge_{\underline{e}}}, \textcircled{\rightarrow_{\underline{e}}}$ satisfy clauses (a)-(f) and (a')-(e') of the definition of DBDA:

(a') Show $|\underline{p}| \textcircled{\ensuremath{\wedge_{\underline{e}}}} |\underline{A}| = |\underline{A}| \textcircled{\ensuremath{\wedge_{\underline{e}}}} |\underline{p}|$:

$$|\underline{p}| \textcircled{\ensuremath{\wedge_{\underline{e}}}} |\underline{A}| = |\underline{p} \wedge_{\underline{e}} \underline{A}|, \text{ by definition}$$

$$= |\underline{A} \wedge_{\underline{e}} \underline{p}|, \text{ by (E ix)}$$

$$= |\underline{A}| \textcircled{\ensuremath{\wedge_{\underline{e}}}} |\underline{p}|, \text{ by definition.}$$

(a) Similar to (a').

(b) Show, e.g., $|\underline{p}| \textcircled{\ensuremath{\vee_{\underline{e}}}} (|\underline{q}| \textcircled{\ensuremath{\vee_{\underline{e}}}} |\underline{A}|) =$

$= (|\underline{p}| \textcircled{\ensuremath{\vee_{\underline{i}}}} |\underline{q}|) \textcircled{\ensuremath{\vee_{\underline{e}}}} |\underline{A}|$:

$$|\underline{p}| \textcircled{\ensuremath{\vee_{\underline{e}}}} (|\underline{q}| \textcircled{\ensuremath{\vee_{\underline{e}}}} |\underline{A}|) = |\underline{p}| \textcircled{\ensuremath{\vee_{\underline{e}}}} (|\underline{q} \vee_{\underline{e}} \underline{A}|), \text{ by def.,}$$

$$= |\underline{p} \vee_{\underline{e}} (\underline{q} \vee_{\underline{e}} \underline{A})|, \text{ by def.,}$$

$$= |(\underline{p} \vee_{\underline{i}} \underline{q}) \vee_{\underline{e}} \underline{A}|, \text{ by a subschema of (Ex'),}$$

$$= (|\underline{p}| \textcircled{\ensuremath{\vee_{\underline{i}}}} |\underline{q}|) \textcircled{\ensuremath{\vee_{\underline{e}}}} |\underline{A}|, \text{ by def.}$$

The rest of (b) can be proved similarly.

(b') Similar to (b).

(c) Show $(|\underline{p}| \textcircled{\ensuremath{\wedge_{\underline{e}}}} |\underline{A}|) \textcircled{\ensuremath{\vee_{\underline{p}}}} |\underline{A}| = |\underline{A}|$:

$$(|\underline{p}| \textcircled{\ensuremath{\wedge_{\underline{e}}}} |\underline{A}|) \textcircled{\ensuremath{\vee_{\underline{p}}}} |\underline{A}| = |\underline{p} \wedge_{\underline{e}} \underline{A}| \textcircled{\ensuremath{\vee_{\underline{p}}}} |\underline{A}|, \text{ by def.,}$$

$$= |(\underline{p} \wedge_{\underline{e}} \underline{A}) \vee_{\underline{p}} \underline{A}|, \text{ by def.,}$$

$$= |\underline{A}|, \text{ by (Exi') and (Eix').}$$

It is of interest, though not strictly necessary, to show that $(|\underline{A}| \circledwedge_e |\underline{p}|) \circledvee_e |\underline{p}| \neq |\underline{p}|$. (If they were equal, then WFP $\cap$ WFI $\neq \emptyset$.) Assume, pro tempore, that they are equal. Now, $(|\underline{A}| \circledwedge_e |\underline{p}|) \circledvee_e |\underline{p}| = |\underline{A} \wedge_e \underline{p}| \circledvee_e |\underline{p}| =$ $= |(\underline{A} \wedge_e \underline{p}) \vee_e \underline{p}|$. If the latter were equal to $|\underline{p}|$, then $(\underline{A} \wedge_e \underline{p}) \vee_e \underline{p} \equiv \underline{p}$, where '$\equiv$' is either '$\equiv_i$' or '$\equiv_p$'. But neither symbol yields a subschema of (Exi') (which is the relevant equivalence in this case, in the form $(\alpha \wedge \beta) \vee \beta \equiv \beta$). Hence, the identities do not hold.

(d)-(f) and (c')-(e') can be proved similarly (and the absence of the other three combinatorially possible equations can be justified, too) ∎

3. The Algebra of DBDAs

We now turn to some purely algebraic results about DBDAs. With one small wrinkle, these are quite analogous to standard results about modules in (universal) algebra.[10]

Let us consider two DBDAs "over" the same Boolean algebra $\underline{B}$: $\langle \underline{M}, \underline{B}, \underline{I}_M, \underline{E}_M \rangle$ and $\langle \underline{N}, \underline{B}, \underline{I}_N, \underline{E}_N \rangle$. Technically, operations such as $\sim_M$ in $\underline{I}_M$ and $\sim_N$ in $\underline{I}_N$ are distinct, but it should not prove too ambiguous in what follows if we write '$\sim$' for both of them (and similarly for the other connectives).

DEF 10: $\underline{h}:\underline{M} \rightarrow \underline{N}$ is a homomorphism from $\underline{M}$ to $\underline{N}$ iff, for all $\underline{m}$, $\underline{m}_1, \underline{m}_2 \in \underline{M}$ and $\underline{b} \in \underline{B}$,

(i) (a) $\underline{h}(\sim\underline{m}) = \sim\underline{h}(\underline{m})$

(b) $\underline{h}(\underline{m}_1 \# \underline{m}_2) = \underline{h}(\underline{m}_1) \# \underline{h}(\underline{m}_2)$

(c) $\underline{h}(\underline{m}_1 \times \underline{m}_2) = \underline{h}(\underline{m}_1) \times \underline{h}(\underline{m}_2)$

(d) $\underline{h}(\underline{m}_1 \supset \underline{m}_2) = \underline{h}(\underline{m}_1) \supset \underline{h}(\underline{m}_2)$, and

(ii) (a) $\underline{h}(\underline{b} \,\underline{+}\, \underline{m}) = \underline{b} \,\underline{+}\, \underline{h}(\underline{m})$ (a') $\underline{h}(\underline{m} \,\underline{+}\, \underline{b}) = \underline{h}(\underline{m}) \,\underline{+}\, \underline{b}$

(b) $\underline{h}(\underline{b} \,\underline{X}\, \underline{m}) = \underline{b} \,\underline{X}\, \underline{h}(\underline{m})$ (b') $\underline{h}(\underline{m} \,\underline{X}\, \underline{b}) = \underline{h}(\underline{m}) \,\underline{X}\, \underline{b}$

(c) $\underline{h}(\underline{b} \underline{\Rightarrow} \underline{m}) = \underline{b} \underline{\Rightarrow} \underline{h}(\underline{m})$ (c') $\underline{h}(\underline{m} \underline{\Rightarrow} \underline{b}) = \underline{h}(\underline{m}) \underline{\Rightarrow} \underline{b}$

DEF 11: Let $\underline{M}' \subseteq \underline{M}$.

Then $\underline{M}'$ is a sub-DBDA of $\underline{M}$ iff, for all $\underline{m}' \in \underline{M}'$ and $\underline{b} \in \underline{B}$,

(i) $\underline{M}'$ is a subalgebra of $\underline{M}$, and

(ii) (a) $\underline{b} \,\underline{+}\, \underline{m}' \in \underline{M}'$ (a') $\underline{m}' \,\underline{+}\, \underline{b} \in \underline{M}'$

(b) $\underline{b} \,\underline{X}\, \underline{m}' \in \underline{M}'$ (b') $\underline{m}' \,\underline{X}\, \underline{b} \in \underline{M}'$

(c) $\underline{b} \underline{\Rightarrow} \underline{m}' \in \underline{M}'$ (c') $\underline{m}' \underline{\Rightarrow} \underline{b} \in \underline{M}'$

DEF 12: Let $\underline{X} \subseteq \underline{M}$.

Then $\underline{N}$ is the sub-DBDA generated by $\underline{X}$ iff $\underline{N}$ is the smallest sub-DBDA of $\underline{M}$ such that $\underline{X} \subseteq \underline{N}$.

We remark that a standard theorem of universal algebra shows that such an $\underline{N}$ exists (cf. Grätzer 1968: 34f).

DEF 13: Let $\underline{X} \subseteq \underline{M}$.

Then $\underline{X}$ generates $\underline{M}$ freely iff

(i) $\underline{X}$ generates $\underline{M}$, and

(ii) for all DBDAs $\underline{N}$ and for all $\varphi:\underline{X} \rightarrow \underline{N}$, there exists a homomorphism $\psi:\underline{M} \rightarrow \underline{N}$ such that, for all $\underline{x} \in \underline{X}$, $\psi(\underline{x}) = \varphi(\underline{x})$.

With these definitions, the following theorem can be proved using standard universal-algebraic techniques (cf. Grätzer 1968: 163):

THM 2: If $\underline{M}$, $\underline{N}$ are free DBDAs with $\underline{n}$ free generators ($\underline{n} \in \omega$), then $\underline{M}$ is isomorphic to $\underline{N}$.

Our next goal is to show that $\mathcal{L}_{\underline{AN}}$ is a free DBDA."

DEF 14: V is a valuation (or interpretation) of AN in $\langle \underline{M}, \underline{B}, \underline{I}, \underline{E} \rangle$ iff V:$\underline{WFN} \rightarrow \underline{M} \cup \underline{B}$ such that

(i) $V(\underline{p}) \in \underline{B}$, $V(\underline{A}) \in \underline{M}$, for all $\underline{p} \in \underline{WFI}$, $\underline{A} \in \underline{WFP}$.

(ii) $V(\neg_{\underline{i}}\underline{p}) = -V(\underline{p})$ $\qquad$ $V(\neg_{\underline{p}}\underline{A}) = \sim V(\underline{A})$

$V(\underline{p} \vee_{\underline{i}} \underline{q}) = V(\underline{p}) + V(\underline{q})$ $\qquad$ $V(\underline{A} \vee_{\underline{p}} \underline{B}) = V(\underline{A}) \# V(\underline{B})$

$V(\underline{p} \wedge_{\underline{i}} \underline{q}) = V(\underline{p}) \cdot V(\underline{q})$ $\qquad$ $V(\underline{A} \wedge_{\underline{p}} \underline{B}) = V(\underline{A}) \times V(\underline{B})$

$V(\underline{p} \rightarrow_{\underline{i}} \underline{q}) = V(\underline{p}) \Rightarrow V(\underline{q})$ $\qquad$ $V(\underline{A} \rightarrow_{\underline{p}} \underline{B}) = V(\underline{A}) \supset V(\underline{B})$

(iii) $V(\underline{p} \vee_{\underline{e}} \underline{A}) = V(\underline{p}) \,\underline{+}\, V(\underline{A})$ $\qquad$ $V(\underline{A} \vee_{\underline{e}} \underline{p}) = V(\underline{A}) \,\underline{+}\, V(\underline{p})$

$V(\underline{p} \wedge_{\underline{e}} \underline{A}) = V(\underline{p}) \,\underline{X}\, V(\underline{A})$ $\qquad$ $V(\underline{A} \wedge_{\underline{e}} \underline{p}) = V(\underline{A}) \,\underline{X}\, V(\underline{p})$

$V(\underline{p} \rightarrow_{\underline{e}} \underline{A}) = V(\underline{p}) \,\underline{\Rightarrow}\, V(\underline{A})$ $\qquad$ $V(\underline{A} \rightarrow_{\underline{e}} \underline{p}) = V(\underline{A}) \,\underline{\Rightarrow}\, V(\underline{p})$

We remark that this is a "Boolean-valued interpretation", since $\underline{B}$ and $\underline{M}$ are Boolean algebras.

Before presenting the next definition, a few heuristic remarks are in order. To mimic precisely the standard interpretation of classical logic in Boolean algebras, the following sort of definition would be used:

Let $\mathcal{K}$ be a class of algebras similar to AN.

Then $\alpha \equiv_{\mathcal{K}} \beta$ iff $V(\alpha) = V(\beta)$ for every valuation V of AN in a member of $\mathcal{K}$.

Then we could say that α "is Boolean-equivalent to" β ($\alpha \equiv_{\underline{B}} \beta$) iff $\mathcal{K}$ = the class of Boolean algebras. An attempt at this point to provide a general soundness theorem for $\underset{\sim}{C}$ (i.e., if $\alpha \equiv \beta$, then $\alpha \equiv_{\underline{B}} \beta$) would fail, since there would be no way to handle equivalences of the form $\underline{p} \equiv_{\underline{e}} \underline{A}$ (where $\underline{p} \equiv_{\underline{e}} \underline{A}$ iff $\underline{p} \leftrightarrow_{\underline{e}} \underline{A}$ is a theorem of $\underset{\sim}{C}$), no provisions having been made in the $\mathcal{K}$-algebras for such equivalences.

In general, too, for the particular applications we have in mind, the $\underline{M}$ and $\underline{B}$ constituents of the DBDAs will have the same cardinality. Were we to identify them, as the clause "$V(\alpha) = V(\beta)$" suggests, we would be destroying our definition of a valuation (clause (i)), or, rather, it would mean that we were interpreting AN in a Boolean algebra, not in a DBDA. Thus, we want a clause like "$V(\alpha) = V(\beta)$", but weaker. We want $V(\alpha)$ to "play the same role" in one algebra that $V(\beta)$ plays in the other (when $\alpha \in$ WFI and $\beta \in$ WFP, say). For example, if $\underline{M}$ and $\underline{B}$ are isomorphic to the 2-element Boolean algebra, viz., $\underset{\sim}{2} = \{1,0\}$, and $\underline{M} \cap \underline{B} = \emptyset$, then we could take $\underline{M} = \{J,N\}$ and $\underline{B} = \{T,F\}$, where J and T "play the same role" as 1.[12] What we need, then is an isomorphism $\underline{h}:\underline{M} \to \underline{B}$ such that $\underline{h}(J) = T$.

Combining these requirements, the definition we need is:

DEF 15: Let $\mathcal{K}$ be a class of algebras similar to AN such that for each $\langle \underline{M},\underline{B},\underline{I},\underline{E}\rangle \in \mathcal{K}$, there is an isomorphism $\underline{h}:\underline{M} \to \underline{B}$ for which $\underline{h}(V(\underline{A})) = V(\underline{p})$ iff $\underline{A} \equiv_{\underline{e}} \underline{p}$ (for all $\underline{A} \in$ WFP, $\underline{p} \in$ WFI, and valuations V of AN in a member of $\mathcal{K}$).

$$\text{Then } \alpha \equiv_{\mathcal{K}} \beta \text{ iff } V(\alpha) = \begin{cases} V(\beta), \text{ if } \alpha,\beta \text{ are both in } \underline{WFI} \text{ or both in } \underline{WFP} \\ \underline{h}(V(\beta)), \text{ if } \alpha \in \underline{WFI} \text{ and } \beta \in \underline{WFP} \\ \underline{h}^{-1}(V(\beta)), \text{ if } \alpha \in \underline{WFP} \text{ and } \beta \in \underline{WFI} \end{cases}$$

for all valuations V of AN in a member of $\mathcal{K}$.

DEF 16: α is Boolean-equivalent to β ($\alpha \equiv_{\underline{B}} \beta$) iff

$\alpha \equiv_{\mathcal{K}} \beta$ and $\mathcal{K}$ = the class of Boolean algebras.

We then have the following results (the proofs are modeled after those in Dunn):

THM 3 (General Soundness for C): If $\alpha \equiv \beta$, then $\alpha \equiv_{\underline{B}} \beta$.

proof (by induction on length of proof):

Suppose $\alpha, \beta \in$ WFI. Then if $\alpha \equiv_{\underline{i}} \beta$ by virtue of one of (Ei), (Eviii-xiv), (Eviii'-xiii'), then $V(\alpha) = V(\beta)$ because of the corresponding postulate for Boolean algebras, viz., DEF 3.ii.a-f, a'-e'. Rules (Eii-vii) preserve Boolean equivalence because of corresponding postulates on identity.

Suppose $\alpha, \beta \in$ WFP. Then $\alpha \equiv_{\underline{p}} \beta$ implies $\alpha \equiv_{\underline{B}} \beta$ analogously.

Finally, if $\alpha \in$ WFI, $\beta \in$ WFP, and $\alpha \equiv_{\underline{e}} \beta$, then $V(\alpha) = \underline{h}(V(\beta))$; and if $\alpha \in$ WFP, $\beta \in$ WFI, and $\alpha \equiv_{\underline{e}} \beta$, then $V(\alpha) = \underline{h}^{-1}(V(\beta))$, analogously ∎

COROLLARY: $\mathcal{L}_{\underline{AN}}$ is a free DBDA.

proof: The set of free generators[13] will be $\underline{X} = \{|\underline{A}| : \underline{A} \in \underline{V}_{\underline{P}}$ & $|\underline{A}| = \{\underline{B} \in \underline{WFP} : \underline{A} \equiv_{\underline{p}} \underline{B}\}\}$. Let $\underline{N}$ be a DBDA and let $\varphi : \underline{X} \to \underline{N}$.

We need to extend φ to a homomorphism $\psi:\mathcal{L}_{\underline{AN}} \to \underline{N}$ such that for all $|\underline{A}| \in \underline{X}$, $\psi(|\underline{A}|) = \varphi(|\underline{A}|)$. To do this, we define a function $V:\underline{WFP} \to \underline{N}$ inductively as follows:

$$V(\underline{A}) =_{\underline{df}} \varphi(|\underline{A}|)$$

$$V(\neg_{\underline{p}}\underline{A}) =_{\underline{df}} \sim V(\underline{A})$$

$$V(\underline{A} \vee_{\underline{p}} \underline{B}) =_{\underline{df}} V(\underline{A}) \# V(\underline{B})$$

$$V(\underline{A} \wedge_{\underline{p}} \underline{B}) =_{\underline{df}} V(\underline{A}) \times V(\underline{B})$$

$$V(\underline{A} \to_{\underline{p}} \underline{B}) =_{\underline{df}} V(\underline{A}) \supset V(\underline{B}).$$

Next, we define $\psi:\mathcal{L}_{\underline{AN}} \to \underline{N}$ such that $\psi(|\underline{A}|) = V(\underline{A})$, for all $\underline{A} \in \underline{WFP}$. It suffices to show that ψ is well-defined and a homomorphism. We shall merely show a representative portion of the proof of the latter:

$$\psi(|\underline{A}| \textcircled{\vee_p} |\underline{B}|) = \psi(|\underline{A} \vee_{\underline{p}} \underline{B}|) = V(\underline{A} \vee_{\underline{p}} \underline{B}) = V(\underline{A}) \# V(\underline{B})$$
$$= \psi(|\underline{A}|) \# \psi(|\underline{B}|).$$

Thus, ψ preserves operations. As for the former, can $|\underline{A}| = |\underline{B}|$ (i.e., can $\underline{A} \equiv_{\underline{p}} \underline{B}$) without $V(\underline{A}) = V(\underline{B})$? Since V is a valuation, then by the soundness theorem just proved, the answer is 'No'; so ψ is well-defined.∎

<u>THM 4 (General Completeness for $\mathcal{C}$)</u>: If $\alpha \equiv_{\underline{B}} \beta$, then $\alpha \equiv \beta$.

<u>proof</u>: Let $V_{\underline{c}}:\underline{WFN} \to \underline{WFI}/_{\equiv_{\underline{i}}} \cup \underline{WFP}/_{\equiv_{\underline{p}}}$ be such that $V_{\underline{c}}(\underline{p}) = |\underline{p}|$ and $V_{\underline{c}}(\underline{A}) = |\underline{A}|$, for all $\underline{A},\underline{p} \in \underline{WFN}$. Then $V_{\underline{c}}$ is the natural homomorphism of $\underline{AN}$ onto $\mathcal{L}_{\underline{AN}}$, and it is a valuation. Assume that $\alpha \not\equiv \beta$. We have 3 cases: $\underline{p} \not\equiv_{\underline{i}} \underline{q}$, $\underline{A} \not\equiv_{\underline{p}} \underline{B}$, $\underline{p} \not\equiv_{\underline{e}} \underline{A}$.

If $p \not\equiv_i q$, then $|p| \neq |q|$; so $V_c(p) \neq V_c(q)$, by definition. Hence, $p \not\equiv_B q$.

If $A \not\equiv_p B$, then $A \not\equiv_B B$, similarly.

If $p \not\equiv_e A$, then $V_c(p) \neq h(V_c(A))$, where h is the isomorphism of DEF 15. Hence, $p \not\equiv_B A$. ■

We conclude this section with a representation theorem for "isomorphic" DBDAs, viz., those DBDAs with which we have been concerned (i.e., those $\langle M,B,I,E\rangle$s for which M is isomorphic to B).

<u>DEF 17</u>: A <u>double Boolean dominance algebra of sets</u> consists of a field of sets $\mathfrak{M}$, a field of sets $\mathfrak{B}$, and operations $\cap^*, \cup^*, \supset^* : (\mathfrak{B} \times \mathfrak{M}) \cup (\mathfrak{M} \times \mathfrak{B}) \rightarrow \mathfrak{M}$ such that

(i) there exists an isomorphism $h: \mathfrak{B} \rightarrow \mathfrak{M}$, and

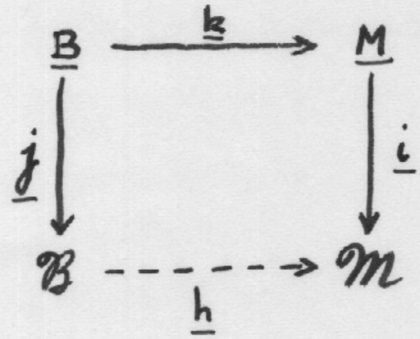
(ii) for all $m \in \mathfrak{M}$, $b \in \mathfrak{B}$,

$$b \cap^* m = h(b) \cap m \qquad m \cap^* b = m \cap h(b)$$
$$b \cup^* m = h(b) \cup m \qquad m \cup^* b = m \cup h(b)$$
$$b \supset^* m = \overline{h(b)} \cup m \qquad m \supset^* b = \overline{m} \cup h(b),$$

where $^{-}$ is the complement operator.

<u>THM 5</u>: Every (isomorphic) DBDA, $\langle M,B,I,E\rangle$, is isomorphic to a DBDA of sets, $\langle \mathfrak{M}, \mathfrak{B}\rangle$.

<u>proof</u>: Since every Boolean algebra is isomorphic to a field of sets, let $i: M \rightarrow \mathfrak{M}$, $j: B \rightarrow \mathfrak{B}$ be such isomorphisms. Let $k: B \rightarrow M$ be an isomorphism. Let $h =_{df} i \circ k \circ j^{-1} : \mathfrak{B} \rightarrow \mathfrak{M}$. Thus, the following diagram commutes:



We need to show that clauses (i) and (ii) of DEF 17 are satisfied. Clearly, $\underline{h}$ is an isomorphism: Let $\underline{b} \in \underline{B}$, $\underline{m} \in \underline{M}$, $\mathfrak{b} \in \mathfrak{B}$, $\mathfrak{m} \in \mathfrak{M}$. Then, e.g., since $\underline{j}^{-1}$, $\underline{k}$, and $\underline{i}$ are isomorphisms,

$$
\begin{aligned}
\underline{h}(\mathfrak{b} \cup \mathfrak{m}) &= \underline{i} \circ \underline{k} \circ \underline{j}^{-1}(\mathfrak{b} \cup \mathfrak{m}) \\
&= \underline{i} \circ \underline{k}(\underline{j}^{-1}(\mathfrak{b}) + \underline{j}^{-1}(\mathfrak{m})), \\
&= \underline{i}(\underline{k} \circ \underline{j}^{-1}(\mathfrak{b}) \# \underline{k} \circ \underline{j}^{-1}(\mathfrak{m})) \\
&= \underline{i} \circ \underline{k} \circ \underline{j}^{-1}(\mathfrak{b}) \cup \underline{i} \circ \underline{k} \circ \underline{j}^{-1}(\mathfrak{m}) \\
&= \underline{h}(\mathfrak{b}) \cup \underline{h}(\mathfrak{m}).
\end{aligned}
$$

To satisfy clause (ii), define $\cap^*$, $\cup^*$, $\supset^*$ such that $\mathfrak{b} \cap^* \mathfrak{m} = \underline{h}(\mathfrak{b}) \cap \mathfrak{m}$, etc.

Thus, $\langle \mathfrak{M}, \mathfrak{B} \rangle$ is a DBDA of sets, and $J{:}\langle \underline{M}, \underline{B}, \underline{I}, \underline{E} \rangle \rightarrow \langle \mathfrak{M}, \mathfrak{B} \rangle$, where $J|_{\underline{B}} = \underline{j}$ and $J|_{\underline{M}} = \underline{i}$, is an isomorphism. ∎

A remark on the final sentence of this proof is perhaps in order. Strictly speaking, $\underline{M}$ is the DBDA and $\mathfrak{M}$ is the DBDA of sets; thus, strictly, $\underline{i}$ is the requisite representation-isomorphism. But to thus focus our attention on $\underline{M}$ (while relegating $\underline{B}$ to the background) is to do <u>practitional</u> logic. To do true <u>noematic</u> logic, we must give equal status to $\underline{M}$ and $\underline{B}$; i.e., we must consider DBDAs to be "systems" of algebras. We now turn to some further observations along these lines.

IV. GENERALIZATIONS AND CONCLUDING REMARKS

In the previous section, we have presented an algebraic interpretation of the "sentential" fragment of Castañeda's deontic logic (cf. n.6). I hope to report on the first-order and modal extensions in future essays. For the former, we will need a dominance analogue of cylindrical algebras; for the latter, wherein the ought-to-do operator is considered, we will need a dominance analogue of modal cylindrical algebras--with a wrinkle: for the ought-to-do operator forms propositions from practitions.[7] Nevertheless, there are some interesting mathematical generalizations and philosophical implications of the work accomplished thus far.

We may extend the notion of a dominance algebra from a pair to an n-tuple:

DEF 18: An n-fold dominance chain of algebras (DCA) is a $(2n+k)$-tuple $\langle A_1,\ldots,A_n,O_1,\ldots,O_n,O^*_1,\ldots,O^*_k\rangle$, where (for each $1\leq i\leq n$) A_i is an algebra with (internal) operations in the set O_i, and (for each $1\leq j\leq k=\frac{n!}{(n-2)!2}$) O^*_j is a set of (external) operations between the A_i such that $O^*_1 \subseteq A_2^{(A_1 x A_2)\cup(A_2 x A_1)}$, $O^*_2 \subseteq A_3^{(A_1 x A_3)\cup(A_3 x A_1)}$, ..., $O^*_n \subseteq A_n^{(A_1 x A_n)\cup(A_n x A_1)}$, $O^*_{n+1} \subseteq A_3^{(A_2 x A_3)\cup(A_3 x A_2)}$, ..., $O^*_k \subseteq A_n^{(A_{n-1} x A_n)\cup(A_n x A_{n-1})}$.

Thus, for any pair of algebras A, B (where A "precedes" B in the DCA), the result of combining a member of A with a member of B

by an external operation is a member of B. It is in this sense that we may say that B "dominates" or "is dominant with respect to" A, and A "is recessive with respect to" B.

We can call an n-fold DCA cyclic iff there are also (external) operations from $(A_n \times A_1) \cup (A_1 \times A_n)$ to A_1. In general, one could "extend" any DCA by adding external "recessive" operations. While such generalizations as these might be of some mathematical interest, they are beyond our present scope.

Now, in view of the closing remark of Section III and in spite of the remark in Section III.1, modules are not clearly dominance algebras (i.e., 2-fold DCAs, with A_1 being a ring and A_2 being the group). Rather, as ordinarily defined (cf. Herstein 1964: 160; but contrast Solian 1977: 30f), they are sets with an "internal" binary operation (specifically, they are groups) and a set of singulary operations (or operators) each of which is of the form: r. (where $r \in A_1$.[14] For our purposes, however, it seems better to regard them as 2-fold DCAs and then to generalize in the direction of DBDAs, since we don't want merely to be talking about the practitional component of C. However, since, for our purposes, the DBDA is only a 2-fold DCA, we can consider it as a more straightforward sort of generalization of a module when convenient (e.g., for proving theorems about free generation, etc.).

To clarify this a bit, there are at least two ways in which to generalize the notion of a module. In the first ("straightforward") way, we might say that a modular algebra is an algebra with internal operations and a set (with some structure) of singulary operators. In the second way, we have a 2-fold DCA, viz., a pair of algebras, each having internal operations and such that there are external operations between the algebras, with one algebra being dominant.[15] This, as we have seen, generalizes still further to *n*-fold DCAs.

Such mathematical generalizations might seem empty.[16] However, there *are* applications of *n*-fold DCAs, where $n>2$. Consider the logic of natural language, which has, besides propositions and practitions, also questions (and commands--cf. n.4). As noted in Section I.3, the combinations of such noemata form a dominance chain. It is unclear which dominates which; perhaps recessive operations would have to be considered. But such noemata as

> If you shouldn't sing, then should you dance?
>
> You should sing, but should you dance?
>
> Sit down, or would you rather dance?
>
> Go, won't you?

suggest that the algebra of well-formed questions dominates that of WFP, which in turn (as we have seen) dominates that of WFI.

Note that we would *not* want to consider this as a (3-fold?) *modular* algebra (whatever that might be), since that would suggest that *questions* are the fundamental units of thought or language. But we need all three (or four, if we count commands). Nor are we interested in questions *primarily*, but in all three (or four) structures equally, together with their interactions. So DCAs (*perhaps* cyclic or even recessive) are (the) appropriate algebraic structures for the mathematical analysis of natural languages.[17]

NOTES

[1]Part II (first-order fragment) and Part III (modal fragment) are forthcoming. See Sect. IV.

[2]I am referring, here, in part to Aristotle's distinctions between practical wisdom vs. scientific knowledge and pure vs. practical syllogisms (cf. Aristotle, Nicomachean Ethics, VI-VII, esp. 1147a30) and in part to Hume's is-ought distinction (cf. Hume 1739: 469 and discussions in Castañeda 1973, 1974: 128ff, 1975: 11ff; and Searle 1964).

[3]Elsewhere, Castañeda has favored a distinction between two modes of predication within the propositional sphere. It thus *appears* that his complete theory requires at least three copulas. Cf. Castañeda 1972b, 1975: 324ff.

[4]There are other noemata; e.g., *mandates* are expressed by imperative sentences, and *questions* are expressed by interrogative sentences. While mandates are discussed at length in Castañeda 1974 and 1975, they do not appear in his formal deontic logic, and so we shall ignore them here. But cf. Sect. IV.

[5]Actions, i.e., *other* than the action of contemplating.

[6]There is a small but interesting terminological difficulty here. In general, I prefer the term 'propositional logic', but that obviously won't do, because of the presence of practitions. 'Noematic logic' would be better but ultimately less perspicuous than the technically incorrect (since practitions are not sentences) 'sentential logic'.

[7]Castañeda makes a distinction between (1) an "ought-to-be" operator which transforms indicative sentences (or propositions) into indicatives (or propositions) thus: ought-to-be(John goes home) = John ought to go home, and (2) an "ought-to-do" operator which transforms practitives (or practitions) into indicatives (or propositions) thus: ought-to-do(John to go home) = John ought to go home. Cf. Castañeda 1972a, 1975: 46, 207; Moore 1903; and Sect. IV below.

[8]Strictly, three different kinds of parentheses: $(_{\underline{i}}$, $(_{\underline{p}}$, $(_{\underline{e}}$, and their right-hand counterparts, can (and perhaps should) be employed; but such distinctions are not crucial to what follows. Alternatively, these distinctions can be made and then uniformly ignored (i.e., tacitly understood) except where ambiguity would threaten.

[9]We could also define $\underline{p} \equiv_{\underline{e}} \underline{A}$ iff $\underline{p} \leftrightarrow_{\underline{e}} \underline{A}$ is a theorem of $\underline{C}$, but this proves to be superfluous for our present purposes. It

close up with p. 31A

is not, however, an idle definition, for some propositions <u>are</u> logically equivalent to some practitions, though they <u>are</u> distinct. Note, too, that the absence of $\equiv_e$ ties in nicely with the absence of the four equations from the definition of DBDA.

[10]Cf., e.g., Hartley and Hawkes 1970, Cohn 1965, and Grätzer 1968.

[11]For those readers losing their way through the forest of connectives, it will be useful to recall DEFs 3 and 4.

[12]In Castañeda's system, J and T are the designated (truth-like) values for practitions and propositions, respectively. Cf. Castañeda 1974: 84, 89, and Ch. 4.

[13]Recall that $\underline{V}_P$ is the set of pure practitive variables; cf. Sect. II.

[14]Thus, where $\underline{m} \in \underline{A}_2$ and $\underline{r}.$ is one of the operators, $\underline{r}.\underline{m} \in \underline{A}_2$.

[15]Another line of generalization is to a <u>2-fold dominant-recessive algebra</u>: a pair of algebras $\langle \underline{A}, \underline{B} \rangle$ together with a set of operations of the form $\underline{d}:(\underline{A} \times \underline{B}) \cup (\underline{B} \times \underline{A}) \rightarrow \underline{B}$ and a set of operations of the form $\underline{r}:(\underline{A} \times \underline{B}) \cup (\underline{B} \times \underline{A}) \rightarrow \underline{A}$. The $\underline{d}$ are <u>dominant</u> operations; the $\underline{r}$ <u>recessive</u> (clearly these terms are arbitrary in this case).

[16]But cf. Solian 1977, Ch. 16, "Multimodules".

[17]Shorter versions of this paper were presented to the Niagara Linguistics Society, the Mathematical Association of America Seaway Section, and the Association for Symbolic Logic. I am grateful to Randall Dipert and J. Michael Dunn for their

(close up with p. 32)

advice and comments, and to the Joint Awards Council/University Awards Committee of the Research Foundation of SUNY for a Faculty Research Fellowship.

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November 21, 1978

Professor William J. Rapaport
Department of Philosophy
State University College
Fredonia, NY 14063

Dear Professor Rapaport:

The referee of your paper, "An Algebraic Interpretation of Deontic Logic, Part I: Sentential Fragment" has now reported. Although I have not verified his criticism of specific points, I agree with his general judgment that the paper largely consists of straightforward uses of familiar mathematical results. With regret I conclude that the paper should not be published in JSL and must therefore decline to accept it.

A photocopy of the referee's report is enclosed. It may be of help to you in planning your future work to get this careful and detailed assessment of this part of it.

I enclose the two copies of the manuscript you have sent. Thank you for enclosing a stamped return envelope.

Sincerely yours,

William Craig

William Craig, Editor
JOURNAL OF SYMBOLIC LOGIC

Enclosures

WC/jm

Referee's report on

An Algebraic Interpretation of Deontic Logic

Part I: Sentential Fragment

The main contribution of this paper is the introduction of the concept of a 'dominance algebra'. These algebras are motivated by problems arising in deontic logic, and hence are of interest to philosophers. However, the paper is mainly mathematical.

The theorems proved about dominance algebras cover only the special case of a 'double Boolean dominance algebra', and even then, all but one concern only a specific DBDA formed from a propositional calculus discussed in the paper. The first result proved about this algebra is just that it is a DBDA. The techniques used in this proof should be familiar to anyone who has studied the Lindenbaum algebra of standard propositional calculus. The paper could be shortened by relegating these straightforward verifications to the reader.

Skipping to the end of the paper, there is a general representation theorem proved for DBDA's. It is an immediate corollary to the Stone representation theorem for Boolean algebras.

The core of the paper is concerned with an algebraic completeness theorem for the calculus considered in the paper. The proofs are all straightforward exercises in working with abstract algebras and equivilance relations defined on them. However, there seems to be a problem with a basic definition which may prevent some of the proofs from being correct.

(First, definition 16 cannot be quite right, because Boolean algebras are not of the correct similarity type to fit the requirements of definition 15. Surely what is intended is that Boolean algebras be considered (in the obvious way) as DBDA's. This is not an important point, but I mention it because the author takes pains to maintain this distinction for special reasons on page 22.)

Definition 15 requires that for any valuation V there is an isomorphism h with the property that $h(V(\underline{A}))=V(\underline{p})$ iff $\underline{A}\equiv_e\underline{p}$. (1)
Let V be the valuation which assigns to every indicative and practitive variable the value $\underline{1}$ of the appropriate Boolean algebra. Since any isomorphism takes 'greatest element' to greatest element, no mapping h can satisfy (1) since this would imply that all pairs of variables (of the appropriate kind) would be provably 'e-equivilant'; i.e. $\underline{A}\equiv_e\underline{p}$, which is absurd. Furthermore, modification of this definition may run into trouble because there do exist Boolean algebras with no non-trivial automorphisms.

Finally, a trivial mistake occurs in the definition of the set $\underline{X}$ on page 23: the clause '$|A|=\{\underline{B}$ WFP : $\underline{A}\equiv_p\underline{B}\}$' is redundant, as it is always satisfied according to its definition(which is all it is) on page 15.

Because of the above criticisms I cannot recommend this paper for publication in the Journal of Symbolic Logic.

State University of New York at Buffalo

DEPARTMENT OF LINGUISTICS

FACULTY OF SOCIAL SCIENCES AND ADMINISTRATION

Buffalo, Dec. 1. 78

A remark concerning your paper "An Algebraic Interpretation of Deontic Logic"

I have read your paper and I think that one can omit the difficulty occurring in definition 15.

You gave the definition of $\leftrightarrow_{\underline{i}}$ (page 13) and wrote "similarly for $\leftrightarrow_{p}$, $\leftrightarrow_{\underline{e}}$". The case of $\leftrightarrow_{p}$ is obvious. But $\leftrightarrow_{\underline{e}}$ can not be defined in a "similar" way. We can not put

$$(*)\quad \underline{P} \leftrightarrow_{\underline{e}} \underline{A} =_{df} (\underline{P} \rightarrow_{\underline{e}} \underline{A}) \wedge_{\underline{e}} (\underline{A} \rightarrow_{\underline{e}} \underline{P})$$

because $\underline{P} \rightarrow_{\underline{e}} \underline{A}$ and $\underline{A} \rightarrow_{\underline{e}} \underline{P}$ are both wfps and $\wedge_{\underline{e}}$ is not applicable to two wfps. The only "similar" (with respect to $\leftrightarrow_{\underline{i}}$ and $\leftrightarrow_{\underline{p}}$) way to define $\leftrightarrow_{\underline{e}}$ is the following:

$$(**)\quad \underline{P} \leftrightarrow_{\underline{e}} \underline{A} =_{df} (\underline{P} \rightarrow_{\underline{e}} \underline{A}) \wedge_{p} (\underline{A} \rightarrow_{\underline{e}} \underline{P}).$$

But $\leftrightarrow_{\underline{e}}$ defined by $(**)$ is not an equivalence relation! (cf. note 9 and definition 15)

Let us go to page 22. According to the above remarks, $\equiv_{\underline{e}}$ can not be defined (in fact, it can not be defined as an equivalence relation). Hence, as far as I can see, we do not need anything like definition 15. The proofs of theorems 3 and 4 work after some obvious corrections. I hope that I did not make any mistake in the above reasoning.

Sincerely yours,

Jurek

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