Annals of Discrete Mathematics 3 (1978) 227–245.

PERCOLATION PROBABILITIES ON THE SQUARE LATTICE

P.D. SEYMOUR and D.J.A. WELSH
Merton College, Oxford, England

1. Introduction

This paper deals mainly with bond percolation on the square lattice. This model is a special but perhaps the most interesting case of the general theory of percolation introduced by Broadbent and Hammersley [4] in 1957. In Section 2 we review briefly the general percolation model; for further details see Frisch and Hammersley [13], Shante and Kirkpatrick [24], Essam [9] or Welsh [29].

In Section 3 we introduce the FKG inequality of Fortuin, Kasteleyn and Ginibre [12]. In Section 4 we introduce the problem of percolation through an $n\times n$ sponge (loosely speaking, when is it possible to move from one side to another of a randomly dammed chessboard?). We examine two of the possible critical probabilities p_T, p_H defined in [29] and use the theory developed for the sponge problem to prove the result

$$p_T+p_H=1.$$

Since Harris [18] has proved $p_H\geq\frac{1}{2}$ and since intuitively one expects the numbers to be equal this suggests that all the critical probabilities for bond percolation on the square lattice have the common value $\frac{1}{2}$.

2. The percolation model

If G is a graph, finite or infinite, we let $V=V(G)$ be its set of vertices and $E=E(G)$ its set of edges. The little graph terminology we use is standard (see for example Berge [2] or Bondy and Murty [3]).

By the *percolation model* on G we mean the assignment of *open* or *closed* to each edge of G with probabilities p and $q=1-p$ respectively, the assignments to be independent for each edge. If an edge is open we picture it as allowing fluid to pass along it; if closed it does not allow fluid to move along it. Thus if A is any subset of edges of the finite graph G, the probability that A is exactly the set of open edges is

$$\pi(A)=p^{|A|}q^{|E\setminus A|}.$$

If Ω denotes the set of all possible assignments, we identify a typical member ω of Ω with the subset of edges which are open in ω. We shall be dealing throughout with a graph G in which $E(G)$ is at most countable and the random variables are on the space Ω. There is never any problem with the measurability or lack of it for the random variables which we shall be discussing and hence we shall usually write X for $X(\omega)$ and so on. For details of similar such arguments see for example [17].

If G is a graph and A, B are subsets of $V(G)$ and U is a subgraph of G,

$$\{A \overset{U}{\rightsquigarrow} B\}$$

denotes the fact that there is a path lying entirely in U which connects some vertex x in A to some vertex y in B. Occasionally we abuse notation and U is not a subgraph of G but just a set of vertices. In such cases we interpret the expression as $\{A \overset{\hat{U}}{\rightsquigarrow} B\}$ where $\hat{U}$ is the graph induced by U.

If Ω is the probability space of the percolation model on G the *event* $\{A \rightsquigarrow B\}$ is the event of Ω that there is some path of open edges linking a vertex of A to a vertex of B.

Throughout $\mathscr{L}$ will denote the square lattice, that is the set of points (x, y) of the plane having integer coordinates x and y and having edges joining each point (x, y) to its nearest neighbours $(x+1, y)$, $(x-1, y)$, $(x, y-1)$, $(x, y+1)$.

As usual in this theory it is convenient to regard $\mathscr{L}$ as the "limit" of a sequence of finite graphs. One suitable sequence is $(\mathscr{L}_n\colon\ 0 \leqslant n < \infty)$ where $\mathscr{L}_n$ is the restriction of $\mathscr{L}$ to the set of vertices $\{(x, y)\colon\ -n \leqslant x \leqslant n,\ -n \leqslant y \leqslant n\}$. $\mathscr{L}$ itself is self-dual; that is, if we consider a new infinite graph $\mathscr{L}^*$ whose vertices are the points $(x+\frac{1}{2}, y+\frac{1}{2})$ where x, y run through the integers, and whose edges are again those lines joining nearest neighbours, then $\mathscr{L}^*$ has the following properties.

(a) It is isomorphic to $\mathscr{L}$.

(b) There is an obvious geometric duality between $\mathscr{L}$ and $\mathscr{L}^*$ inasmuch as they can be drawn as geometric duals in the plane, see for example [3].

Almost exclusively in this paper we shall restrict ourselves to percolation on $\mathscr{L}$, or some sequence of subgraphs of $\mathscr{L}$ which approach $\mathscr{L}$.

Suppose we now regard the origin O as a source of fluid. We say that a point v of $\mathscr{L}$ is *wet* by fluid from the origin if there is a path consisting of open edges from O to v, and otherwise v is *dry*.

Let us now fix $\boldsymbol{p}$, $0 \leqslant \boldsymbol{p} \leqslant 1$. We let $\boldsymbol{P}_n(\boldsymbol{p})$ be the probability that at least n points are wet by fluid from the origin. Clearly

$$\boldsymbol{P}_n(\boldsymbol{p}) \geqslant \boldsymbol{P}_{n+1}(\boldsymbol{p})$$

so that

$$\boldsymbol{P}(\boldsymbol{p}) = \lim_{n\to\infty} \boldsymbol{P}_n(\boldsymbol{p})$$

exists, and satisfies

$$0 \leqslant \boldsymbol{P}(\boldsymbol{p}) \leqslant 1.$$

However, though each $\boldsymbol{P}_n(\boldsymbol{p})$ is a polynomial in $\boldsymbol{p}$ and can be calculated, it still is not known for example whether or not $\boldsymbol{P}(\boldsymbol{p})$ is continuous in $\boldsymbol{p}$. Broadbent and Hammersley [4] show that there exists a *critical probability* $\boldsymbol{p}_{\mathrm{H}}$ defined by

$$\boldsymbol{p}_{\mathrm{H}} = \inf \boldsymbol{p} : \boldsymbol{P}(\boldsymbol{p}) > 0.$$

Harris [18] proved that

$$\boldsymbol{p}_{\mathrm{H}} \geqslant \frac{1}{2} \tag{1}$$

and Hammersley [16] that

$$\boldsymbol{p}_{\mathrm{H}} \leqslant 0.646790. \tag{2}$$

As pointed out in [29] there are several other "critical probabilities" in the literature, and the relationships among them are obscure to say the least. First consider $V(\boldsymbol{p})$, the expected number of points wet by the source at the origin—that is,

$$V(\boldsymbol{p}) = \sum_{n \geqslant 1} \boldsymbol{P}_n(\boldsymbol{p}).$$

We define $\boldsymbol{p}_{\mathrm{T}}$ by

$$\boldsymbol{p}_{\mathrm{T}} = \inf \boldsymbol{p} : V(\boldsymbol{p}) = \infty.$$

Since $V(\boldsymbol{p})$ is infinite if $\boldsymbol{P}(\boldsymbol{p}) > 0$, we have immediately that

$$\boldsymbol{p}_{\mathrm{T}} \leqslant \boldsymbol{p}_{\mathrm{H}}. \tag{3}$$

One of our results below will be that

$$\boldsymbol{p}_{\mathrm{T}} \leqslant \frac{1}{2}. \tag{4}$$

This has an easy proof, but also follows from our main theorem:

Theorem 2.1. *In percolation on the square lattice the critical probabilities $\boldsymbol{p}_{\mathrm{T}}$, $\boldsymbol{p}_{\mathrm{H}}$ satisfy $\boldsymbol{p}_{\mathrm{T}} + \boldsymbol{p}_{\mathrm{H}} = 1$.*

Our proof of this is quite long and is given in Section 5. Thus on an intuitive level at least there is strong evidence to support the following conjecture.

Conjecture 2.2. $p_T = p_H = \frac{1}{2}$.

We should emphasize that for several years there has been a folklore belief that the above conjecture was proved by Sykes and Essam [25] in 1964. Sykes and Essam in fact show that under certain (as yet unproven) assumptions a *third* quantity p_E associated with percolation on the square lattice is equal to $\frac{1}{2}$. Although various attempts have been made (see for example Grimmett [14]) to prove that the assumptions demanded by Sykes and Essam are correct it is a much more difficult (in fact, as far as we can see, hopelessly intractable) problem to relate p_E with p_H or p_T. Even the very definition of p_E is shrouded with mystery.

3. The FKG inequality

In 1971 Fortuin, Kasteleyn and Ginibre [12] proved a remarkable inequality showing that non-decreasing functions on a finite distributive lattice are positively correlated by all positive measures which have a certain convexity property. This inequality was originally applied to Ising ferromagnets in an arbitrary magnetic field, but as pointed out in [12] it is also closely related to a lemma used by Harris [18] in proving Theorem 2.1. In [23] we showed that the inequality has diverse applications in combinatorial theory, and Kempermann [20] has given some new applications in probability theory. In this section we shall use it to obtain some new results in percolation, first passage percolation, and random graph theory. It is also used repeatedly in the proof of our main result in Section 5.

Two random variables X and Y are *covariant* if $\mathscr{E}(XY) \geq (\mathscr{E}X)(\mathscr{E}Y)$. Two events A, B are *covariant* if their respective indicator functions are covariant. Clearly (if $\boldsymbol{P}(B) \neq 0$) A, B are covariant if and only if

$$\boldsymbol{P}(A \mid B) \geq \boldsymbol{P}(A).$$

A set $\{X_1, \ldots, X_k\}$ of random variables is *covariant* if for any subset $I \subseteq \{1, \ldots, k\}$, $\mathscr{E}(\prod_{i \in I} X_i) \geq \prod_{i \in I} \mathscr{E}(X_i)$.

Let D be a distributive lattice, where obviously we are using "lattice" in its algebraic sense. A function $f: D \to R$ is called *increasing* if $f(x) \leq f(y)$ for any pair of elements x, y of D such that $x \leq y$. A function f is *decreasing* if $-f$ is increasing.

When D is finite and $\mu: D \to R^+$, the *μ-average* of a function $f: D \to R$ is given by

$$\langle f \rangle = \langle f \rangle_\mu = \Big(\sum_{x \in D} f(x) \mu(x) \Big) \Big/ \sum_{x \in D} \mu(x).$$

The original version of the FKG inequality proved in [12] is as follows.

Theorem 3.1 (The FKG inequality). *Let D be a finite distributive lattice and let $\mu : D \to R^+$ satisfy*

$$\mu(x)\mu(y) \leqslant \mu(x \wedge y)\mu(x \vee y) \quad (x, y \in D). \tag{5}$$

Then if f, g are both increasing or both decreasing functions, then

$$\langle fg \rangle \geqslant \langle f \rangle \langle g \rangle. \tag{6}$$

An obvious corollary of this is that if f and g are functions on D which are monotone but in the opposite sense, then

$$\langle fg \rangle \leqslant \langle f \rangle \langle g \rangle.$$

Before proceeding to give some applications of Theorem 3.1 we prove a lemma. The proof is elementary, but we give it because we use the result several times later.

Lemma 3.2. *If A_1, A_2 are covariant events in Ω with $\boldsymbol{P}(A_1) = \boldsymbol{P}(A_2)$ then*

$$\boldsymbol{P}(A_1) \geqslant 1 - [1 - \boldsymbol{P}(A_1 \cup A_2)]^{1/2}$$

Proof.

$$\begin{aligned} \boldsymbol{P}(A_1 \cup A_2) &= \boldsymbol{P}(A_1) + \boldsymbol{P}(A_2 \mid \Omega \backslash A_1)\boldsymbol{P}(\Omega \backslash A_1) \\ &\leqslant \boldsymbol{P}(A_1) + \boldsymbol{P}(A_2)[1 - \boldsymbol{P}(A_1)], \end{aligned}$$

since A_1, A_2 are covariant. Hence since $\boldsymbol{P}(A_1) = \boldsymbol{P}(A_2)$ we have

$$1 - \boldsymbol{P}(A_1 \cup A_2) \geqslant [1 - \boldsymbol{P}(A_1)]^2,$$

which completes the proof.

Example 3.3 (Random graphs). For each positive integer n let D_n be the lattice of subsets of E_n, the set of edges of the complete graph K_n. Now let μ be defined as

$$\mu A = \boldsymbol{p}^{|A|}\boldsymbol{q}^{|E \backslash A|}.$$

Consider the following events about the random graphs ω on n vertices in which each edge of K_n exists or does not exist with probabilities $\boldsymbol{p}, 1 - \boldsymbol{p}$;

A: ω is planar,
B: ω is hamiltonian,
C: ω is 4-colourable.

It is clear that whereas A and C have decreasing indicator functions, B has an increasing indicator function. Hence the FKG inequality gives such statements as

$$\begin{aligned} &\boldsymbol{P}[\text{random graph } \omega \text{ is hamiltonian} \mid \omega \text{ is planar}] \\ &\quad \leqslant \boldsymbol{P}[\text{random graph } \omega \text{ is hamiltonian}]. \end{aligned} \tag{7}$$

$$\boldsymbol{P}[\text{random graph } \omega \text{ is 4 colourable} \mid \omega \text{ is hamiltonian}] \leq \boldsymbol{P}[\text{random graph } \omega \text{ is 4 colourable}]. \tag{8}$$

Although intuitively appealing, such results do not seem easy to prove directly and serve to indicate the power of the FKG inequality.

Now the reader will notice that in the FKG inequality as stated in Theorem 3.1 the lattice D is restricted to being finite. Various infinite extensions of the inequality and of a stronger result of Holley [19] have been made recently by Batty [1], Cartier [5], Edwards [7], Kempermann [20] and Preston [22]. However, as far as the main theorems of this paper are concerned the only infinite extension we need is the following covariance inequality first proved by Fortuin [10].

Theorem 3.4. *Let G be a countable graph and let $\boldsymbol{P}$ be the probability measure induced by a percolation model on G. Let f and g be increasing functions on the partially ordered probability space associated with this model. Then if $\mathscr{E}$ is the expectation operator associated with $\boldsymbol{P}$,*

$$\mathscr{E}(fg) \geq \mathscr{E}(f)\mathscr{E}(g)$$

whenever the expectations exist.

Immediately from this we see that the results obtained in Example 3.3 above hold when G is a countably infinite graph.

We close this section by sketching a proof of an extension of Harris' correlation result to first passage percolation theory as defined by Hammersley and Welsh [17]. One interest of this extension is that Theorem 3.5 below was the original "physical result" which motivated Batty's infinite extension [1] of the FKG inequality.

Let G be a (finite or countably infinite) graph directed or undirected, with vertex set V and edge set E. Suppose that to each edge e_i of G we assign a random variable u_i drawn, independently for each edge, from a distribution $F(x)$. We call u_i the *time coordinate* of e_i.

The set Ω of E-tuples ω, defined by $\omega(e_i) = u_i$, $e_i \in E$, is called the *phase space* and can be ordered by

$$\omega \leq \omega' \Leftrightarrow \omega(e_i) \leq \omega'(e_i) \quad \forall e_1 \in E.$$

If x, y are any two vertices of G we write $t_{xy}(\omega)$ to denote the *first passage (shortest) time* between x and y over paths of G, when it is in state ω. More precisely

$$t_{xy}(\omega) = \inf t(P, \omega),$$

where $t(P, \omega)$ is the sum of the time coordinates of the edges making up the path P, and the infimum is over all paths P joining x and y.

Now for any points x_1, x_2, y_1, y_2 of $V(G)$ it is obvious that $t_{x_1x_2}(\omega)$ and $t_{y_1y_2}(\omega)$

are monotone on Ω, in the sense that

$$\omega \leq \omega' \Rightarrow t_{x_1x_2}(\omega) \leq t_{x_1x_2}(\omega').$$

Thus we can apply the infinite version of the FKG inequality implicit in the work of Batty [1] and Edwards [7] to get the result that the pair of random variables $t_{x_1x_2}$ and $t_{y_1y_2}$ are covariant. More generally, if A, B are two subsets of V and

$$t_{AB}(\omega) = \inf_{\substack{x \in A \\ y \in B}} t_{xy}(\omega)$$

represents the first passage time between A and B when G is in state ω we have the following general result:

Theorem 3.5. *For any sets A, B, C, D of vertices of the countable graph G the first passage times t_{AB} and t_{CD} are covariant random variables.*

4. The sponge problem

In this section we consider a new variant of the percolation problem. It is of some interest in its own right; indeed we studied it purely for its own sake before realising that it was a useful tool in giving insight into the relationship between p_T and p_H. Most of the results of this section will be used in proving our main result, Theorem 2.1. The vertex or atom percolation version of this problem has also been studied numerically by Kurkijarvi and Padmore [21]. However, they assume as physically obvious certain results which we have found impossible to prove rigorously.

The $m \times n$ *sponge* consists of the subgraph $T(m, n)$ of $\mathscr{L}$ induced on the mn points

$$\{(x, y)\colon 1 \leq x \leq n, 1 \leq y \leq m\}.$$

Each of the m points $(1, y)$, $1 \leq y \leq m$, is regarded as an infinite source of fluid which may percolate through those edges of the sponge which are open. The probability that any edge is open is p, independently for each edge.

We let $S_p(m, n) = S(m, n)$ denote the probability that some of the points (n, k), $1 \leq k \leq m$, become wet by fluid.

Trivial inequalities are

$$S(m, n+1) \leq S(m, n), \tag{9}$$

$$S(m, n) \leq S(m+1, n). \tag{10}$$

A basic, but extremely useful, result is the following.

Theorem 4.1. *For all* p, $0 \leqslant p \leqslant 1$, *and all positive integers* $m \geqslant 1$, $n \geqslant 2$,

$$S_p(m, n) + S_q(n-1, m+1) = 1,$$

where $q = 1 - p$.

Proof. Construct a new graph $G(m, n)$ from the $m \times n$ sponge $T(m, n)$ as follows. Identify all the vertices $(1, y)$, $1 \leqslant y \leqslant m$, in a new vertex x_1. (Remove all edges which become loops.) Similarly identify all vertices (n, y), $1 \leqslant y \leqslant m$, in a vertex x_2. Add a new edge e joining x_1 and x_2. The graph $G(m, n)$ is planar, and its planar dual G^* is isomorphic to $G(n-1, m+1)$. Now consider any assignment ω of open and closed values to the edges of $T(m, n)$. There is a path of open edges from one of the vertices $(1, y)$, $1 \leqslant y \leqslant m$, to one of (n, y), $1 \leqslant y \leqslant m$, if and only if there is a cycle in $G(m, n)$ consisting of e and otherwise edges which are open in ω. But, by the elementary max-flow min-cut theorem, either there is such a cycle in $G(m, n)$, or there is a cycle in G^* consisting of e and otherwise edges closed in ω (and not both). But since G^* is isomorphic to $G(n-1, m+1)$, and an edge of $T(m, n)$ is closed with probability q, the result follows.

Hence if we define

$$S_n(p) = S_p(n, n+1),$$

we have for all positive integers n,

$$S_n(p) + S_n(1-p) = 1 \quad (0 \leqslant p \leqslant 1).$$

In particular

$$S_n(\tfrac{1}{2}) = \tfrac{1}{2} \quad (1 \leqslant n < \infty). \tag{11}$$

It is also clear that $S_n(p)$ is a monotonic increasing function of p, satisfying for all n,

$$S_n(0) = 0, \qquad S_n(1) = 1.$$

However we have *not* been able to prove:

Conjecture 4.2. *For all* p, $0 \leqslant p \leqslant 1$, $\lim S_n(p)$ *exists.* (*We have shown that, even if the limit always exists, it is not continuous.*)

Conjecture 4.3. *For* $p < \frac{1}{2}$ (*respectively* $> \frac{1}{2}$), $S_n(p)$ *is a monotone decreasing* (*respectively increasing*) *function of* n.

We now relate $S_n(p)$ with $\boldsymbol{P}_n(p)$.

Theorem 4.4. *For any positive integer* n *and* $0 \leqslant p \leqslant 1$

$$S_n(p) \leqslant 1 - (1 - \boldsymbol{P}_{n+1}(p))^n.$$

Proof. Consider the $n \times (n+1)$ sponge and let

$$X = \{(x, y) : x = 1, 1 \leq y \leq n\},$$
$$Y = \{(x, y) : x = n+1, 1 \leq y \leq n\}.$$

Then

$$1 - S_n(\boldsymbol{p}) = \boldsymbol{P}(X \overset{T_n}{\nrightarrow} Y)$$
$$\geq \boldsymbol{P}\left(\bigcap_{i=1}^{n} A_i\right),$$

where $A_i = \{(1, i) \overset{\mathscr{L}}{\nrightarrow} Y\}$, $1 \leq i \leq n$.

But by the FKG inequality the A_i are covariant events, each having probability $\geq 1 - \boldsymbol{P}_{n+1}(\boldsymbol{p})$. Hence

$$1 - S_n(\boldsymbol{p}) \geq (1 - \boldsymbol{P}_{n+1}(\boldsymbol{p}))^n$$

and the result follows.

Suppose now we define the *critical sponge probability* $\boldsymbol{p}_s$ by

$$\boldsymbol{p}_s = \inf \boldsymbol{p} : \limsup_{n \to \infty} S_n(\boldsymbol{p}) > 0.$$

Then we know from (11) that

$$\boldsymbol{p}_s \leq \tfrac{1}{2}. \tag{12}$$

It will follow from the proof of Theorem 2.1 that

$$0.353210 \leq \boldsymbol{p}_T \leq \boldsymbol{p}_s. \tag{13}$$

One final result which we need before proving the main theorem is the following: For any n,

$$S(n, n) \leq 8S(n-1, n-1). \tag{14}$$

To see this consider the $n \times n$ sponge. If there is a path across it, then there must be a path across one of the four $(n-1) \times (n-1)$ sponges inside it or there must be a path from the top to the bottom of one of these sponges. Considering the union of these events gives (14).

5. Proof of Theorem 2.1

We shall prove Theorem 2.1 by the series of Lemmas 5.1–5.6 below. However, it is probably instructive to show the broad outline here.

First in Lemma 5.1, which is relatively straightforward, we show that

$$\boldsymbol{p}_T + \boldsymbol{p}_H \leq 1. \tag{15}$$

In Lemmas 5.2–5.4 we prove various inequalities about the $S(m, n)$ which enable us to show in Lemma 5.5 that if $\boldsymbol{p} < 1 - \boldsymbol{p}_{\mathrm{H}}$ not only does the sequence $S_n(\boldsymbol{p})$ converge, but

$$\lim_{n\to\infty} S_n(\boldsymbol{p}) = 0.$$

In Lemma 5.6, we show that if $\boldsymbol{p} > \boldsymbol{p}_{\mathrm{T}}$,

$$\limsup_{n\to\infty} S_n(\boldsymbol{p}) \geqslant \delta > 0.$$

Thus we have $1 - \boldsymbol{p}_{\mathrm{H}} \leqslant \boldsymbol{p}_{\mathrm{T}}$ which with (15) proves our final result that

$$\boldsymbol{p}_{\mathrm{T}} + \boldsymbol{p}_{\mathrm{H}} = 1.$$

Lemma 5.1. $\boldsymbol{p}_{\mathrm{T}} + \boldsymbol{p}_{\mathrm{H}} \leqslant 1.$

Proof. Let L be the set of points $\{(i, 0): i \geqslant 0\}$ of $\mathscr{L}$ and for $1 \leqslant n < \infty$ let L_n be the set of points $\{(-i, 0): i \geqslant n\}$. We choose a fixed $\boldsymbol{p} < \boldsymbol{p}_{\mathrm{T}}$; and then

$$\sum_{i=1}^{\infty} \boldsymbol{P}_i(\boldsymbol{p}) < \infty.$$

Choose N so that $\sum_{i \geqslant N} \boldsymbol{P}_i(\boldsymbol{p}) < 1$. Now

$$\boldsymbol{P}\{(-i, 0) \overset{\mathscr{L}}{\rightsquigarrow} L\} = \boldsymbol{P}\{(0, 0) \overset{\mathscr{L}}{\rightsquigarrow} L_i\} \leqslant \boldsymbol{P}_i(\boldsymbol{p}).$$

Hence

$$\sum_{i \geqslant N} \boldsymbol{P}((-i, 0) \overset{\mathscr{L}}{\rightsquigarrow} L\} < 1.$$

Hence

$$\boldsymbol{P}\{L_N \overset{\mathscr{L}}{\rightsquigarrow} L\} < 1.$$

Now let B_i $(1 \leqslant i < \infty)$ be the points $(-i + \frac{1}{2}, \frac{1}{2})$ $(0 \leqslant i < \infty)$ which are the vertices of the dual lattice $\mathscr{L}^*$. For each assignment ω of open and closed to the edges of $\mathscr{L}$ we will consider $\mathscr{L}^*$ in state ω^*, where if e is closed in $\mathscr{L}$ under ω then the corresponding edge e^* of $\mathscr{L}^*$ is closed in $\mathscr{L}^*$.

Let $B^* = B^*(\omega)$ be the set of points of the dual lattice which are joined by a path of closed edges of $\mathscr{L}^*$ to one of $B_1, \ldots, B_N$. Suppose that we assume that with probability one B^* is finite. Then if B^* is finite let P^* be those edges of $\mathscr{L}^*$ joining vertices of B^* to vertices of $\mathscr{L}^* \backslash B^*$. Then every edge in P^* must be open in $\mathscr{L}^*$.

Now since B^* is finite P^* must cut L_N and must also cut L. Hence by elementary graph theory arguments there is an open path in $\mathscr{L}$ connecting L_N with L. But we have chosen N so that the event $L_N \rightsquigarrow L$ has probability strictly less than 1. Hence the assumption that B^* is finite with probability one is false and we must have

$$\boldsymbol{P}(|B^*| = \infty) > 0.$$

But if for $1 \leqslant i \leqslant N$ we let

$$A_i = \{\omega\colon B_i \text{ is connected in } \mathscr{L}^* \text{ by a closed path to an infinite number of points of } \mathscr{L}^*\},$$

then

$$\boldsymbol{P}(|B^*| = \infty) \leqslant \sum_{i=1}^{N} \boldsymbol{P}(A_i).$$

Since $\boldsymbol{P}(A_i) = \boldsymbol{P}(\boldsymbol{q})$, we must have

$$N\boldsymbol{P}(\boldsymbol{q}) > 0,$$

which implies

$$\boldsymbol{q} \geqslant \boldsymbol{p}_{\mathrm{H}}$$

so that $\boldsymbol{p}_{\mathrm{T}} + \boldsymbol{p}_{\mathrm{H}} \leqslant 1$ as required.

Lemma 5.2. *If* $S(2n, 2n) = \tau$, *then* $S(2n, 4n)) \geqslant \tau(1-(1-\tau)^{1/2})^8$.

Proof. Consider the following regions of the square lattice.

$$R = \{(x, y)\colon 1 \leqslant x \leqslant 4n,\ 1 \leqslant y \leqslant 2n\},$$
$$X = \{(x, y)\colon x = 1,\ 1 \leqslant y \leqslant 2n\},$$
$$Z = \{(x, y)\colon x = 2n,\ 1 \leqslant y \leqslant 2n\},$$
$$W = \{(x, y)\colon x = n+1,\ 1 \leqslant y \leqslant 2n\},$$
$$W_1 = \{(x, y)\colon x = n+1,\ 1 \leqslant y \leqslant n\},$$
$$W_2 = \{(x, y)\colon x = n+1,\ n+1 \leqslant y \leqslant 2n\},$$
$$U_1 = \{(x, y)\colon n+1 \leqslant x \leqslant 3n,\ y = 1\},$$
$$U_2 = \{(x, y)\colon n+1 \leqslant x \leqslant 3n,\ y = 2n\},$$
$$S_1 = \{(x, y)\colon 1 \leqslant x \leqslant 2n,\ 1 \leqslant y \leqslant 2n\},$$
$$S = \{(x, y)\colon n+1 \leqslant x \leqslant 3n,\ 1 \leqslant y \leqslant 2n\}.$$

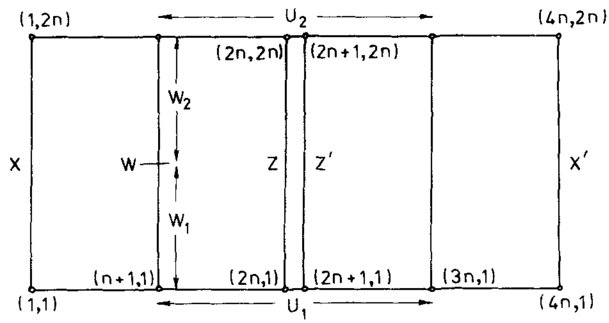


Fig. 1.

We illustrate the situation in Fig. 1.

Now for any subset of vertices A of $\mathscr{L}$ let A' be defined by

$$A' = \{(4n+1-x,\ y) \colon (x, y) \in A\}.$$

so that for example

$$W' = \{(x, y)\colon x = 3n,\ 1 \leqslant y \leqslant 2n\},$$
$$X' = \{(x, y)\colon x = 4n,\ 1 \leqslant y \leqslant 2n\},$$
$$S_1' = \{(x, y)\colon 2n+1 \leqslant x \leqslant 4n,\ 1 \leqslant y \leqslant 2n\}.$$

Consider now the events A_1, A_2, A_3 of Ω defined by

$$A_1 = \{\omega\colon W \overset{S}{\rightsquigarrow} W'\}$$

$A_2 = \{\omega$: there is an open path from X to Z in S_1 which meets an open path from U_1 to U_2 in $S\}$,

$A_3 = \{\omega$: there is an open path from X' to Z' in S_1' which meets an open path from U_1 to U_2 in $S\}$.

Then since A_1, A_2, A_3 are monotone in the same sense they are covariant and since also

$$A_1 \cap A_2 \cap A_3 \subseteq \{X \overset{R}{\sim} X'\},$$

we have

$$\begin{aligned} S(2n, 4n) &\geqslant \boldsymbol{P}(A_1 \cap A_2 \cap A_2) \\ &\geqslant \boldsymbol{P}(A_1)(\boldsymbol{P}(A_2))^2 \\ &= S(2n, 2n)(\boldsymbol{P}(A_2))^2. \end{aligned}$$

We now consider $\boldsymbol{P}(A_2)$. We wish to show

$$\boldsymbol{P}(A_2) \geqslant (1-(1-\tau)^{1/2})^4.$$

Let $(P_i\colon 1 \leqslant i \leqslant k)$ be the collection of paths in S_1 which join X to Z and which have the additional property that their last point Q_i of intersection with W is a point of W_1. For $1 \leqslant i \leqslant k$ let F_i be the section of P_i from Q_i to Z. Then each F_i is a path from W_i to Z.

Let X_i be the event that there is an open path in S from F_i to U_2 which uses only one vertex of F_i and no vertex of F_i'. Let X_i' be the event that there is an open path in S from F_i' to U_2 which uses only one vertex of F_i' and no vertex of F_i.

Now the set of points $F_i \cup F_i'$ separates U_1 from U_2 in S. Hence if there is an open path in S from U_1 to U_2 then either X_i or X_i' occurs. Hence

$$\begin{aligned} \boldsymbol{P}(X_i \cup X_i') &\geqslant \boldsymbol{P}(U_1 \overset{S}{\rightsquigarrow} U_2) \\ &= S(2n, 2n) = \tau. \end{aligned}$$

But X_i, X_i' are covariant, and by symmetry have equal probabilities; hence by Lemma 3.2,

$$\boldsymbol{P}(X_i) = \boldsymbol{P}(X_i') \geqslant 1 - \sqrt{(1-\tau)}.$$

Let us now fix i and consider the three events,

$$\begin{aligned} B_1 &= B_1^{(i)} = \{\omega\colon \text{path } P_i \text{ is open}\}, \\ B_2 &= B_2^{(i)} = \{\omega\colon \text{for each } j \neq i \text{ such that } P_j \text{ lies in the} \\ &\qquad\qquad \text{region bounded by } P_i \text{ and } y = 1,\ P_j \text{ is not open}\}, \\ B_3 &= B_3^{(i)} = X_i. \end{aligned}$$

We assert

$$\boldsymbol{P}(B_1 \cap B_2 \cap B_3) \geqslant (1 - \sqrt{(1-\tau)})\boldsymbol{P}(B_1 \cap B_2).$$

For

$$\boldsymbol{P}(B_1 \cap B_2 \cap B_3) = \boldsymbol{P}(B_2 \cap B_3 \mid B_1)\boldsymbol{P}(B_1),$$

and if B_1 occurs, then the occurrence of B_2 depends only on the state of the edges of $\mathscr{L}$ strictly below P_i in S_1, and the occurrence of B_3 depends only on the state of edges strictly above $F_i \cup F_i'$ in S. Since these two sets of edges are disjoint

$$\boldsymbol{P}(B_2 \cap B_3 \mid B_1) = \boldsymbol{P}(B_2 \mid B_1)\boldsymbol{P}(B_3 \mid B_1).$$

Hence

$$\begin{aligned} \boldsymbol{P}(B_1 \cap B_2 \cap B_3) &= \boldsymbol{P}(B_2 \mid B_1)\boldsymbol{P}(B_3 \mid B_1)\boldsymbol{P}(B_1) \\ &= \boldsymbol{P}(B_2 \cap B_1)\boldsymbol{P}(B_3 \mid B_1). \end{aligned}$$

But since B_1 and B_3 are covariant

$$\boldsymbol{P}(B_3 \mid B_1) \geqslant \boldsymbol{P}(B_3) = \boldsymbol{P}(X_i) \geqslant 1 - \sqrt{(1-\tau)}.$$

Hence

$$\boldsymbol{P}(B_1 \cap B_2 \cap B_3) \geqslant (1 - \sqrt{(1-\tau)})\boldsymbol{P}(B_1 \cap B_2).$$

Now note that

$$\boldsymbol{P}((B_1^{(1)} \cap B_2^{(1)}) \cup (B_1^{(2)} \cap B_2^{(2)}) \cup \cdots \cup (B_1^{(k)} \cap B_2^{(k)}))$$
$$= \sum_1^k \boldsymbol{P}(B_1^{(i)} \cap B_2^{(i)}).$$

But

$$\boldsymbol{P}((B_1^{(1)} \cap B_2^{(1)}) \cup \cdots \cup (B_1^{(k)} \cap B_2^{(k)})) = \boldsymbol{P}(B_1^{(1)} \cup \cdots \cup B_1^{(k)}),$$

and

$$\begin{aligned}\boldsymbol{P}\Big(\bigcup_i (B_1^{(i)} \cap B_3^{(i)})\Big) &\geq \sum_{i=1}^k \boldsymbol{P}(B_1^{(i)} \cap B_2^{(i)} \cap B_3^{(i)}) \\ &\geq (1-\sqrt{(1-\tau)}) \sum_{i=1}^k \boldsymbol{P}(B_1^{(i)} \cap B_2^{(i)}) \\ &= (1-\sqrt{(1-\tau)})\boldsymbol{P}(\text{at least one } P_i \text{ is open}).\end{aligned}$$

Now consider the event C that at least one of the P_i is open. Let $\{\hat{P}_i: 1 \leq i \leq k\}$ be the collection of paths in S_1 which join X to Z and which have the property that their last point of intersection with W is a point of W_2.

The event $\hat{C}$ that at least one of the $\hat{P}_i$ is open is covariant with C and by symmetry

$$\boldsymbol{P}(\hat{C}) = \boldsymbol{P}(C).$$

Also $\boldsymbol{P}(\hat{C} \cup C) = S(2n, 2n) = \tau$ so that by Lemma 5.1,

$$\boldsymbol{P}(C) \geq 1-\sqrt{(1-\tau)},$$

and

$$\boldsymbol{P}(B_1^{(i)} \cap B_3^{(i)} \text{ for some } i) \geq (1-\sqrt{(1-\tau)})^2 \tag{16}$$

Let E_1 be the event that there is a point $w \in W_1$ such that

$$w \overset{S_1}{\rightsquigarrow} X, \qquad w \overset{S}{\rightsquigarrow} U_2, \qquad w \overset{S}{\rightsquigarrow} Z.$$

Let E_2 be the event that there is a point $v \in W_2$ such that

$$v \overset{S_1}{\rightsquigarrow} X, \qquad v \overset{S}{\rightsquigarrow} U_1, \qquad v \overset{S}{\rightsquigarrow} Z.$$

Now

$$E_1 \cap E_2 \subseteq A_2$$

and hence the proof of Lemma 5.2 is complete if we show that

$$\boldsymbol{P}(E_1 \cap E_2) \geq (1-\sqrt{(1-\tau)})^4.$$

But E_1, E_2 are covariant and by symmetry

$$\boldsymbol{P}(E_1) = \boldsymbol{P}(E_2).$$

Hence it is enough to show that

$$\boldsymbol{P}(E_1) \geqslant (1 - \sqrt{(1-\tau)})^2.$$

But (by drawing a picture) E_1 occurs if, for some i, P_i is open and F_i is joined to U_2 by an open path in S. That is

$$\boldsymbol{P}(E_1) \geqslant \boldsymbol{P}\Big(\bigcup_i (B_1^{(i)} \cap B_3^{(i)})\Big).$$

Thus with (16) we have the required result.

Lemma 5.3. $S(2n, 6n) \geqslant [S(2n, 2n)]^3 \,(1 - \sqrt{(1 - S(2n, 2n))})^{16}$

Proof. Consider the following regions of $\mathscr{L}$ (see Fig. 2):

$U = \{(x, y)\colon y = 2n, 2n+1 \leqslant x \leqslant 4n\}$,
$V = \{(x, y)\colon y = 1, 2n+1 \leqslant x \leqslant 4n\}$,
$S = \{(x, y)\colon 2n+1 \leqslant x \leqslant 4n, 1 \leqslant y \leqslant 2n\}$,
$R = \{(x, y)\colon 1 \leqslant x \leqslant 6n, 1 \leqslant y \leqslant 2n\}$,
$X = \{x, y)\colon x = 1, 1 \leqslant y \leqslant 2n\}$,
$Z = \{(x, y)\colon x = 2n, 1 \leqslant y \leqslant 2n\}$,
$W = \{(x, y)\colon x = 4n, 1 \leqslant y \leqslant 2n\}$,
$Y = \{(x, y)\colon x = 6n, 1 \leqslant y \leqslant 2n\}$.

Let

$$A = \{\omega\colon X \overset{R}{\rightsquigarrow} W\},$$

$$B = \{\omega\colon Z \overset{R}{\rightsquigarrow} Y\},$$

$$C = \{\omega\colon U \overset{S}{\rightsquigarrow} V\}.$$

Then

$$A \cap B \cap C \subseteq \{\omega\colon X \overset{R}{\rightsquigarrow} Y\},$$

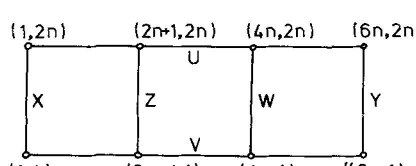


Fig. 2.

and since A, B, C are monotone in the same sense and hence covariant we have

$$\begin{aligned} \boldsymbol{P}(X \overset{R}{\rightsquigarrow} Y) &\geq \boldsymbol{P}(A)\boldsymbol{P}(B)\boldsymbol{P}(C) \\ &= [S(2n, 4n)]^2 S(2n, 2n). \end{aligned}$$

which with Lemma 5.2 proves the result.

Let $R(n)$ be the annulus of the square lattice $\mathscr{L}$ shown in Fig. 2 bounded by the squares C_n, D_n, where C_n consists of the lines

$$y = -3n + 1, \quad x = 3n, \quad y = 3n, \quad x = -3n + 1$$

and D_n consists of the lines

$$y = -n, \quad x = n + 1, \quad y = n + 1, \quad x = -n.$$

Lemma 5.4. *The probability that there is an open cycle around the annulus $R(n)$, that is a cycle of open edges encircling the square D_n and encircled by C_n, is at least*

$$S(2n, 2n)^{12}(1 - \sqrt{(1 - S(2n, 2n))})^{64}.$$

Proof. Let A, B, C, D be the regions of $R(n)$ defined as follows (see Fig. 3):

$$\begin{aligned} A &= \{(x, y)\colon -3n + 1 \leq x \leq -n, -3n + 1 \leq y \leq 3n\}, \\ B &= \{(x, y)\colon -3n + 1 \leq x \leq 3n, -3n + 1 \leq y \leq -n\}, \\ C &= \{(x, y)\colon n + 1 \leq x \leq 3n, -3n + 1 \leq y \leq 3n\}, \\ D &= \{(x, y)\colon -3n + 1 \leq x \leq 3n, n + 1 \leq y \leq 3n\}. \end{aligned}$$

Let

$$\begin{aligned} X &= \{(x, y)\colon -3n + 1 \leq x \leq -n, y = -3n + 1\}, \\ X' &= \{(x, y)\colon -3n + 1 \leq x \leq -n, y = 3n\}, \\ Y &= \{(x, y)\colon x = -3n + 1, n + 1 \leq y \leq 3n\}, \\ Y' &= \{(x, y)\colon x = 3n, n + 1 \leq y \leq 3n\}, \\ U &= \{(x, y)\colon x = -3n + 1, -3n + 1 \leq y \leq -n\}, \\ U' &= \{(x, y)\colon x = 3n, -3n + 1 \leq y \leq -n\}, \\ W &= \{(x, y)\colon n + 1 \leq x \leq 3n, y = -3n + 1\}, \\ W' &= \{(x, y)\colon n + 1 \leq x \leq 3n, y = 3n\}. \end{aligned}$$

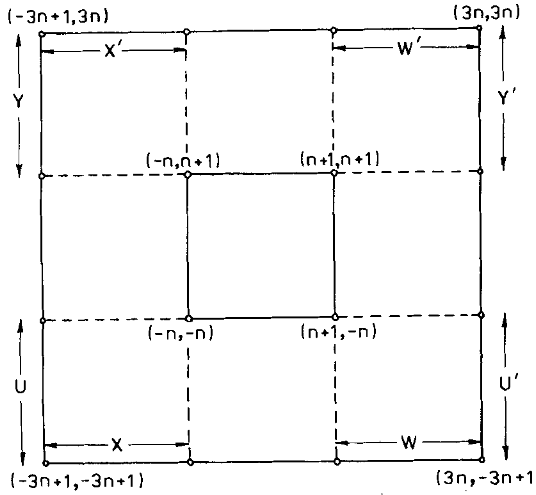
Then if F_n is the event that there is an open cycle around $R(n)$ we have

$$F_n \supseteq \{X \overset{A}{\rightsquigarrow} X'\} \cap \{U \overset{B}{\rightsquigarrow} U'\} \cap \{W \overset{C}{\rightsquigarrow} W'\} \cap \{Y \overset{D}{\rightsquigarrow} Y'\}.$$

Now the events on the right hand side are monotone in the same sense and thus covariant and each has probability $S(2n, 6n)$, so that

$$\boldsymbol{P}(F_n) \geq (S(2n, 6n))^4$$

which with Lemma 5.3 proves Lemma 5.4.



Lemma 5.5. *If* $p<1-p_H$, *then* $\lim_{n\to\infty} S(n, n)$ *exists and is zero, in other words*

$$p_s \geq 1-p_H.$$

Proof. If $p<1-p_H$, then $q>p_H$ so that there is a positive probability of an infinite closed path from the origin in the dual lattice $\mathscr{L}^*$. Suppose that for some $\epsilon>0$, $S(n, n)>8\epsilon$ for infinitely many n. Choose n_1, $n_2, \ldots$ so that $R(n_1)$, $R(n_2), \ldots$, are disjoint annuli and $S(2n_i, 2n_i)>\epsilon$ for each i. This is possible by (14).

Now by Lemma 5.4, the probability that there is an open path around $R(n_i)$ is at least

$$\epsilon^{12}(1-\sqrt{(1-\epsilon)})^{64}$$

for each i. Since the disjointness of the $R(n_i)$ makes these events independent, the Borel–Cantelli lemmas imply that with probability one there can be no closed infinite path from the origin in $\mathscr{L}^*$. Thus we have a contradiction.

Lemma 5.6. *If* $\epsilon>0$ *and* $p\geq p_T$, *then for infinitely many values of* n,

$$(1-S(2n, 2n))^{12}(1-\sqrt{S(2n, 2n)})^{64} \leq \tfrac{8}{9}+\epsilon.$$

In other words if $p\geq p_T$, *then* $\limsup_{n\to\infty} S(2n, 2n)\geq \delta$, *where* δ *is a little bit bigger than* 5×10^{-6}.

Proof. Suppose the lemma is false, and choose N so that for all $n\geq 3^N$ the inequality fails. Now by Theorem 4.1 if $q=1-p$,

$$S_q(2n, 2n)\geq S_q(2n-1, 2n+1)=1-S_p(2n, 2n).$$

Hence by Lemma 5.4 if $3^t \geqslant 3^N$ then the probability that there is a closed cycle around the annulus $R(3^t)$ is at least $\frac{8}{9}+\epsilon$.

Hence by the duality theory of graphs the probability of the event D_t that in $\mathscr{L}^*$ there is an open path from the origin through $R(3^t)$ is not more than $\frac{1}{9}-\epsilon$.

If the number of points in $\mathscr{L}^*$ which are wet by a source at the origin is N_p^*, then we have

$$\begin{aligned}\mathscr{E}(N_p^*) &\leqslant 4\times 3^{2N} + \sum_{t\geqslant N} |R(3^t)| \prod_{N\leqslant k\leqslant t-1} \boldsymbol{P}(D_k)\\ &\leqslant 4\times 3^{2N} + \sum_{t\geqslant N} 4\times 3^{2N}(\tfrac{1}{9}-\epsilon)^{n-N}\\ &<\infty\end{aligned}$$

which contradicts $\boldsymbol{p} \geqslant \boldsymbol{p}_{\mathrm{T}}$.

As our final corollary note that from Lemma 5.6 we know that

$$\boldsymbol{p} > \boldsymbol{p}_{\mathrm{T}} \Rightarrow \limsup_{n\to\infty} S_n(\boldsymbol{p}) \geqslant \delta > 0,$$

whereas if $\boldsymbol{p} < \boldsymbol{p}_{\mathrm{T}}$ then

$$\lim_{n\to\infty} S_n(\boldsymbol{p}) = 0.$$

Hence we have shown that even if Conjecture 4.2 is true and $\lim_{n\to\infty} S_n(\boldsymbol{p})$ exists and equals $S(\boldsymbol{p})$ say, then $S(\boldsymbol{p})$ must be a discontinuous function of $\boldsymbol{p}$.

Note also that our proof gives the result (13) and we can sum up the situation with the set of inequalities

$$0.353210 \leqslant \boldsymbol{p}_{\mathrm{T}} \leqslant \boldsymbol{p}_{\mathrm{s}} \leqslant \tfrac{1}{2} \leqslant \boldsymbol{p}_{\mathrm{H}} \leqslant 0.646790 \tag{17}$$

which with Lemma 5.5 imply that for $\boldsymbol{p} > 0.646790$,

$$\lim_{n\to\infty} S_n(\boldsymbol{p}) = 1. \tag{18}$$

Acknowledgement

We would like to thank R.T. Smythe for pointing out an error in an earlier version of the paper, and G.R. Grimmett for some helpful comments.

References

[1] C.J.K. Batty, An extension of an inequality of R. Holley, Quart. J. Math. Oxford. Ser, 27 (1976) 457–461.

[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).
[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976).
[4] S.R. Broadbent and J.M. Hammersley, Percolation processes I. Crystals and mazes, Proc. Cambridge Philos. Soc. 53 (1957) 626–641.
[5] P. Cartier, Inégalités de correlation en mécanique statistique, Seminaire Bourbaki 25 (1973) 401–423.
[6] C. Domb, Some statistical problems connected with crystal lattices, J. Roy. Statist. Soc. 26 (1964) 367–397.
[7] D.A. Edwards, On the Holley–Preston inequalities, Proc. Edinburgh. Math. Soc., to appear.
[8] J. W. Essam, Graph theory and statistical physics, Discrete Math. 1 (1971) 83–112.
[9] J.W. Essam, Percolation and cluster size, in: C. Domb and M.S. Green, eds., Phase Transition and Critical Phenomena 2 (1972) 197–270.
[10] C.M. Fortuin, On the random cluster model. II. The percolation model, Physica 58 (1972) 393–418.
[11] C.M. Fortuin and P.W. Kasteleyn, On the random cluster model I. Introduction and relation to other models, Physica 57 (1972) 536–564.
[12] C.M. Fortuin, P.W. Kasteleyn and J. Ginibre, Correlation inequalities on some partially ordered sets, Comm. Math. Phys. 22 (1971) 89–103.
[13] H.L. Frisch and J.M. Hammersley, Percolation processes and related topics, SIAM J. 11 (1963) 894–918.
[14] G.R. Grimmett, On the number of clusters in the percolation model, J. London Math. Soc. 2 (1976) 346–350.
[15] J.M. Hammersley, Percolation processes: lower bounds for the critical probability, Ann. Math. Statist. 28 (1957) 790–795.
[16] J.M. Hammersley, Bornes superieures de la probabilité critique dans un processus de filtration, Proc. 87th Int. Colloq. CNRS (Paris, 1957) 17–37.
[17] J.M. Hammersley and D.J.A. Welsh, First passage percolation, subadditive processes, stochastic networks and generalised renewal theory. in: Bernoulli (1723) Bayes (1763) Laplace (1813) Anniversary Volume, University of California Berkeley (1963) (Springer, Berlin, 1965) 61–110.
[18] T.E. Harris, A lower bound for the critical probability in a certain percolation process, Proc. Cambridge Philos. Soc. 56 (1960) 13–20.
[19] R. Holley, Remarks on the FKG inequalities. Comm. Math. Phys. 36 (1974) 227–231.
[20] J.H.B. Kemperman, On the FKG-inequality for measures on a partially ordered space, (to appear).
[21] J. Kurkijarvi and T.C. Padmore, Percolation in two-dimensional lattices, J. Phys. A 8 (1975) 683–696.
[22] C.J. Preston, A generalization of the FKG inequalities, Comm. Math. Phys. 36 (1974) 233–241.
[23] P.D. Seymour and D.J.A. Welsh, Combinatorial applications of an inequality from statistical mechanics, Math. Proc. Cambridge Philos. Soc. 77 (1975) 485–495.
[24] V.K.S. Shante and S. Kirkpatrick, An introduction to percolation theory, Adv. Phys. 20 (1971) 325–356.
[25] M.F. Sykes and J.W. Essam, Exact critical percolation probabilities for the site and bond problems in two dimensions, J. Math. Phys. 5 (1964) 1117–1132.
[26] M.F. Sykes, D.S. Gaunt and M. Glen, Percolation processes in two dimensions II–IV, J. Phys. A. Math. Gen. 9 (1976) 97–103, 715–724, 725–730.
[27] H.N.V. Temperley and E.H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem, Proc. Roy. Soc. London Ser. A 322 (1971) 251–280.
[28] W.T. Tutte, A ring in graph theory, Proc. Cambridge Philos. Soc. 43 (1947) 26–40.
[29] D.J.A. Welsh, Percolation and related topics, Science Progress 64 (1977) 67–85.