Refuting Conj that Someone Gets Equal Shares

35-13

1 \( f(35, 13) = \frac{64}{143} \) AND nobody can get all shares the same size

Erik Metz disproved the conjecture that for all \( m, s \) someone gets shares of all the same size. Below is that proof. Ken Regan made the conjecture and I was hoping it would be true.

We try to keep notation down to a minimum since Ken is a muffin-newbie. We do use the following terminology.

1. A 5-share is a share that goes to a student who gets 5 shares. Same for 6.

2. A 5-student is a student who gets 5 shares. Same for 6.

3. A shares buddy is the other share from the muffin it came from. Note that a share of size \( x \) has a buddy of size \( 1 - x \).

We prove that ANY protocol for \( f(35, 13) \) DOES NOT have a student get all shares the same size. In passing we show that \( f(35, 13) \leq \frac{35}{13} \). We prove more than we need since I thought it would be helpful to derive a protocol. It was not. We already HAVE a protocol; however, I was hoping to gain insights into how to generate one by computer. Oh Well. On the bright side I think that generating protocols by computer, the MUFFIN2 project, is going well. \( f(35, 13) \) will be a good litmus test for it.

2 Any Protocol for \( f(35, 13) \) must...

Theorem 2.1

1. In any protocol that shows \( f(35, 13) \geq \frac{64}{143} \):

   (a) There are eight 5-students.

   (b) There are five 6-students.
(c) All 6-shares are in \([\frac{64}{143}, \frac{65}{143}]\). There are 30 of them.

(d) All 5-shares are in \([\frac{69}{143}, \frac{74}{143}]\) (called \(S5\)- Small 5-shares) or \([\frac{78}{143}, \frac{79}{143}]\) (called \(L5\)-Large 5 shares). There are 10 \(S5\)'s and 30 \(L5\)'s.

(e) The following diagram summarizes what must happen.

\[
\begin{array}{ccccccccc}
\text{64}_{143} & \text{65}_{143} & \text{66}_{143} & \text{67}_{143} & \text{68}_{143} & \text{69}_{143} & \text{70}_{143} & \text{71}_{143} & \text{72}_{143} \\
\text{30 6-shs} & 0 \text{ shs} & 10 \text{ S5 shs} & 0 \text{ shs} & 30 \text{ L5 shs} & 16_{44} \\
\end{array}
\]

2. In any protocol that shows \(f(35, 13) > \frac{64}{143}\) we have the exact same points as above with one change:

\[
\begin{array}{ccccccccc}
\text{64}_{143} & \text{65}_{143} & \text{66}_{143} & \text{67}_{143} & \text{68}_{143} & \text{69}_{143} & \text{70}_{143} & \text{71}_{143} & \text{72}_{143} \\
\text{30 6-shs} & 0 \text{ shs} & 10 \text{ S5 shs} & 0 \text{ shs} & 30 \text{ L5 shs} & 16_{44} \\
\end{array}
\]

(We omit the proof as its essentially identical to that of Part 1.)

Proof:

Assume we have a protocol that shows \(f(35, 13) \geq \frac{64}{143}\).

Case 1: Some student gets \(\geq 7\) shares. Some piece is \(\leq \frac{35}{13 \times 7} = \frac{5}{13} < \frac{64}{143}\).

Case 2: Some student gets \(\leq 4\) shares. Some student gets a share of size \(\geq \frac{35}{13 \times 4} = \frac{35}{52}\). The other part of the muffin it came from is of size \(\leq 1 - \frac{35}{52} = \frac{17}{52} < \frac{64}{143}\).

Case 3: Every muffin is cut in 2 shares and every student gets either 5 or 6 shares. Note that the total number of shares is 70 Let \(s_5\) (\(s_6\)) be the number of 4-students (5-students). From
\[5s_5 + 6s_6 = 70\]
\[s_5 + s_6 = 13\]

\[s_5 = 8, \ s_6 = 5\]

**Case 3.1:** There is a 6-share \( < \frac{65}{143} \). Take it away to the person who has it. Now he has \( \frac{35}{13} - \frac{65}{143} = \frac{320}{143} \) divided in 5 shares. One of those shares is \( < \frac{320}{5 \times 143} = \frac{64}{143} \).

**Case 3.2:** There is a 5-share \( < \frac{69}{143} \). Take it away to the person who has it. Now he has \( \frac{35}{13} - \frac{69}{143} = \frac{316}{143} \) divided in 4 shares. One of those shares is \( > \frac{79}{143} \). The muffin it came from, the other piece is \( < 1 - \frac{79}{143} = \frac{64}{143} \).

**Case 3.3:** There is a 6-share \( < \frac{64}{143} \). This one is self-evident.

**Case 3.4:** There is a 5-share \( > \frac{79}{143} \). The muffin it came from, the other piece is \( < 1 - \frac{79}{143} = \frac{64}{143} \).

**Case 3.5:** All 5-shares are in \( \left[ \frac{64}{143}, \frac{65}{143} \right] \) and all 6-shares are in \( \left[ \frac{69}{143}, \frac{79}{143} \right] \).

The following picture captures what we know so far.

\[
\begin{array}{cccc}
\frac{64}{143} & \text{6-shares} & \frac{65}{143} & \text{No Shares} & \frac{69}{143} & \text{5-shares} & \frac{79}{143} \\
\end{array}
\]

Claim 1: There are no shares \( x \in \left( \frac{74}{143}, \frac{78}{143} \right) \).

**Proof of Claim 1** If there was such a share then the other part of the muffin it came from is in \( \left[ \frac{65}{143}, \frac{69}{143} \right] \). By the premise of Case 3.5 there are no such shares.

**End of Proof of Claim 1**

The following picture captures what we know so far where \( S5 \) stands for Small 5-shares, and \( L5 \) stands for Large 5-shares.

\[
\begin{array}{cccc}
\frac{64}{143} & \text{6-shs} & \frac{65}{143} & \text{No shs} & \frac{69}{143} & S5 \text{ shs} & \frac{74}{143} & \text{No shs} & \frac{78}{143} & L5 \text{ shs} & \frac{79}{44} \\
\end{array}
\]
Recall that $s_5 = 8$, $s_6 = 5$.

**Claim 2:**

1. The number of 6-shares equals the number of $L_5$ shares.

2. There are 30 6-shares, 30 $L_5$-shares, and 10 $S_5$-shares. (This follows from Part 1.)

**Proof of Claim 2:**

1) If $x$ is a 6-share then $x \in (\frac{64}{143}, \frac{65}{143})$. hence its buddy is a 6-share in $(\frac{78}{143}, \frac{79}{143})$. Therefore buddy is a bijection of 6-shares to $L_5$-shares.

2) There are $s_5 = 3$ 4-students, hence there are 12 5-shares. There are $s_6 = 2$ 5-students, hence there are 10 6-shares. But by part 1 the number of 5-shares and 6-shares is the same. Hence this cannot occur.

**End of Proof of Claim 2**

Hence we have:

\[
\begin{bmatrix}
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\frac{64}{143} & 30 \text{ 6-shs} & \frac{65}{143} & 0 \text{ shs} & \frac{69}{143} & 10 \text{ 5-shs} & \frac{74}{143} & 0 \text{ shs} & \frac{78}{143} & 30 \text{ L5 shs} & \frac{79}{143}
\end{bmatrix}
\]

3 **No Protocol for** $f(35, 13) \geq \frac{35}{13}$ **has a Same Size Shares Student**

We prove more than we need in the next theorem. I was hoping it would help later, but it did not.

**Theorem 3.1**

1. In a protocol that shows $f(35, 13) \geq \frac{64}{143}$ no student gets $\geq 5$ shares in $(\frac{64}{143}, \frac{65}{143}]$.

2. In a protocol that shows $f(35, 13) \geq \frac{64}{143}$ every 6-students gets $\geq 2$ shares of size $\frac{64}{143}$.

(This follows from Part 1.)
3. \( f(35, 13) \leq \frac{64}{143} \). (This follows from Part 2.)

**Proof:** Assume, by way of contradiction, that some student gets \( \geq 5 \) shares in \( (\frac{64}{143}, \frac{65}{143}) \).

*Note:* Many of the Claims we need in this proof do not use the premise that that some student gets \( \geq 5 \) shares in \( (\frac{64}{143}, \frac{65}{143}) \). When this happens we note it for clarity and sanity.

These are all 6-shares so it must be a 6-student. So this 6-student uses \( \leq 1 \) share of size \( \frac{64}{143} \).

There are 5 more students and they use the remaining 24 6-shares.

**Claim 1 and its Proof**

1. A 6-student gets \( \leq 5 \) shares of size \( \frac{64}{143} \). **Proof:** if he used 6 then he would have
   \[ 6 \times \frac{64}{143} < \frac{35}{13}. \] (Did not use premise.)

2. There are \( \leq 21 \) shares of size \( \frac{64}{143} \). **Proof:** By the above and Part 1 (a) one of the 6-students gets \( \leq 1 \) such shares, and (b) the remaining four of the 6-students get \( \leq 5 \) such shares.

3. There are \( \leq 21 \) shares of size \( \frac{79}{143} \). **Proof:** These are the buddies of the shares of size \( \frac{64}{143} \).

We now look at the 5-people and 5-shares. There are eight 5-people and they will use \( \leq 21 \) shares of size \( \frac{79}{143} \), 10 shares in \( \left[ \frac{69}{143}, \frac{74}{143} \right] \), and of course other shares.

**Claim 2: and Proof**

1. No 5-student gets \( \geq 3 \) S5 shares. **Proof:** \( \leq 3 \times \frac{74}{143} + 2 \times \frac{79}{143} < \frac{35}{13} \). (Did not use premise.)

2. At least two 5-students get 2 S5 shares and 3 L5 shares. **Proof:** There are eight 5-students. There are 10 S5-shares. By Part 1 no 5-student gets \( \geq 3 \) S5 shares. Hence at least two 5-students get 2 S5 shares. (Did not use premise.)

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3. At least two 5-students get two $\frac{74}{143}$’s and three $\frac{79}{143}$’s. **Proof:** The only way a 5-student can get up to $\frac{35}{13}$ using 2 S5 shares and 3 L5 shares is two $\frac{74}{143}$’s and three $\frac{79}{143}$’s. (Did not use Premise)

4. There are at least four $\frac{74}{143}$ shares **Proof:** From last item. (Did not use Premise)

5. There are at least four $\frac{69}{143}$ shares **Proof:** The buddies of the at least four $\frac{74}{143}$. (Did not use Premise)

**RECAP:**

1. There are eight 5-students. There are $\leq 21$ shares of size $\frac{79}{143}$. There are two 5-students that have two $\frac{74}{143}$’s and three $\frac{79}{143}$’s. So, removing those students and shares we have six 5-students and $\leq 15$ shares of size $\frac{79}{143}$.

2. There are $\geq 4$ shares of size $\frac{69}{143}$ that still have not been used.

3. SO- we have six 5-students who have to use $\leq 15$ shares of size $\frac{79}{143}$ and $\geq 4$ shares of size $\frac{69}{143}$.

**Claim 3 and Proof:**

1. No student gets 2 $\frac{69}{143}$’s. **Proof:** $2\frac{69}{143} + 3\frac{79}{143} < \frac{35}{13}$. (Did not use Premise.)

2. At least four 6-students get exactly 1 $\frac{69}{143}$. **Proof:** There are 6 students who have to use 4 $\frac{69}{143}$. From Part 1 nobody can use two of them. (Did not use Premise.)

3. A student who has a $\frac{69}{143}$ has to have four $\frac{79}{143}$. **Proof:** $\frac{69}{143} + 4 \times \frac{79}{143} = \frac{35}{13}$. So need to use all four of the top possible piece. (Did not use Premise.)

4. There are four 5-students who each have 4 $\frac{79}{143}$ shares. **Proof:** From the points above. (Did not use Premise.)

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RECAP: Let's inventory the 5-students

Two 5-students use 3 shares of size \( \frac{79}{143} \).

Four 5-students use 4 shares of size \( \frac{79}{143} \).

Two 5-students we have not looked at.

The 5-students use \( 2 \times 3 + 4 \times 4 = 22 \) shares of size \( \frac{79}{143} \).

Too bad- there are only 21 of them.

Contradiction.

\[ \textbf{Theorem 3.2} \quad \text{In a protocol that shows } f(35, 13) \geq \frac{64}{143}, \text{ no student gets same size shares.} \]

Proof:

There are two cases: a 5-student gets same-size-shares and a 6-student gets same-size-shares.

1) If a 5-student has shares all the same size then they are of size \( \frac{35}{5 \times 13} = \frac{7}{13} = \frac{77}{143} \). By the diagram above, there are no shares of size \( \frac{77}{143} \).

2) If a 6-student has shares all the same size then they are of size \( \frac{35}{6 \times 13} \). By calculation one can show

\[
\frac{64}{143} < \frac{35}{6 \times 13} < \frac{65}{143}
\]

Hence there is a student who gets \( \geq 5 \) (in fact, 6) shares in \(( \frac{64}{143}, \frac{65}{143} \)). This contradicts Theorem 3.1.