Refuting Conj that Someone Gets Equal Shares

35-13

1 $f(35,13)=\frac{64}{143}$ AND nobody can get all shares the same size

Erik Metz disproved the conjecture that for all m,s someone gets shares of all the same size. Below is that proof. Ken Regan made the conjecture and I was hoping it would be true.

We try to keep notation down to a minimum since Ken is a muffin-newbie. We do use the following terminology.

1. A *5-share* is a share that goes to a student who gets 5 shares. Same for 6.

2. A *5-student* is a student who gets 5 shares. Same for 6.

3. A shares *buddy* is the other share from the muffin it came from. Note that a share of size x has a buddy of size $1-x$.

We prove that ANY protocol for $f(35,13)$ DOES NOT have a student get all shares the same size. In passing we show that $f(35,13)\le\frac{35}{13}$. We prove more than we need since I thought it would be helpful to derive a protocol. It was not. We already HAVE a protocol; however, I was hoping to gain insights into how to generate one by computer. Oh Well. On the bright side I think that generating protocols by computer, the MUFFIN2 project, is going well. $f(35,13)$ will be a good litmus test for it.

2 Any Protocol for $f(35,13)$ must...

Theorem 2.1

1. *In any protocol that shows $f(35,13)\ge\frac{64}{143}$:*

   (a) *There are eight 5-students.*

   (b) *There are five6-students.*

(c) *All 6-shares are in* $[\frac{64}{143}, \frac{65}{143}]$. *There are 30 of them.*

(d) *All 5-shares are in* $[\frac{69}{143}, \frac{74}{143}]$ *(called S5- Small 5-shares) or* $[\frac{78}{143}, \frac{79}{143}]$ *(called L5- Large 5 shares). There are 10 S5's and 30 L5's.*

(e) *The following diagram summarizes what must happen.*

[	$---$	](	$---$	)[	$---$	](	$---$	)[	$---$	]
$\frac{64}{143}$	*30 6-shs*	$\frac{65}{143}$	*0 shs*	$\frac{69}{143}$	*10 S5 shs*	$\frac{74}{143}$	*0 shs*	$\frac{78}{143}$	*30 L5 shs*	$\frac{79}{44}$

2. *In any protocol that shows* $f(35,13) > \frac{64}{143}$ *we have the exact same points as above with one change:*

(	$---$	)[	$---$	](	$---$	)[	$---$	](	$---$	)
$\frac{64}{143}$	*30 6-shs*	$\frac{65}{143}$	*0 shs*	$\frac{69}{143}$	*10 S5 shs*	$\frac{74}{143}$	*0 shs*	$\frac{78}{143}$	*30 L5 shs*	$\frac{79}{44}$

(We omit the proof as its essentially identical to that of Part 1.)

Proof:

Assume we have a protocol that shows $f(35,13) \geq \frac{64}{143}$.

Case 1: Some student gets ≥ 7 shares. Some piece is $\leq \frac{35}{13\times 7} = \frac{5}{13} < \frac{64}{143}$.

Case 2: Some student gets ≤ 4 shares. Some student gets a share of size $\geq \frac{35}{13\times 4} = \frac{35}{52}$. The other part of the muffin it came from is of size $\leq 1 - \frac{35}{52} = \frac{17}{52} < \frac{64}{143}$.

Case 3: Every muffin is cut in 2 shares and every student gets either 5 or 6 shares. Note that the total number of shares is 70 Let s_5 (s_6) be the number of 4-students (5-students). From

$$\begin{aligned} 5s_5 + 6s_6 &= 70 \\ s_5 + s_6 &= 13 \end{aligned}$$

$s_5 = 8$, $s_6 = 5$

Case 3.1: There is a 6-share $< \frac{65}{143}$. Take it away to the person who has it. Now he has $> \frac{35}{13} - \frac{65}{143} = \frac{320}{143}$ divided in 5 shares. One of those shares is $< \frac{320}{5 \times 143} = \frac{64}{143}$.

Case 3.2: There is a 5-share $< \frac{69}{143}$. Take it away to the person who has it. Now he has $> \frac{35}{13} - \frac{69}{143} = \frac{316}{143}$ divided in 4 shares. One of those shares is $> \frac{79}{143}$. The muffin it came from, the other piece is $< 1 - \frac{79}{143} = \frac{64}{143}$.

Case 3.3: There is a 6-share $< \frac{64}{143}$. This one is self-evident.

Case 3.4: There is a 5-share $> \frac{79}{143}$. The muffin it came from, the other piece is $< 1 - \frac{79}{143} = \frac{64}{143}$.

Case 3.5: All 5-shares are in $[\frac{64}{143}, \frac{65}{143}]$ and all 6-shares are in $[\frac{69}{143}, \frac{79}{143}]$.

The following picture captures what we know so far.

[	− − − − − −	]	− − − − − −	[	− − − − −	]
$\frac{64}{143}$	6-shares	$\frac{65}{143}$	No Shares	$\frac{69}{143}$	5-shares	$\frac{79}{143}$

Claim 1: There are no shares $x \in (\frac{74}{143}, \frac{78}{143})$.

Proof of Claim 1 If there was such a share then the other part of the muffin it came from is in $[\frac{65}{143}, \frac{69}{143}]$. By the premise of Case 3.5 there are no such shares.

End of Proof of Claim 1

The following picture captures what we know so far where $S5$ stands for Small 5-shares, and $L5$ stands for Large 5-shares.

[	− − −	](	− − −	)[	− − −	](	− − −	)[	− − −	]
$\frac{64}{143}$	6-shs	$\frac{65}{143}$	No shs	$\frac{69}{143}$	S5 shs	$\frac{74}{143}$	No shs	$\frac{78}{143}$	L5 shs	$\frac{79}{44}$

Recall that $s_5 = 8$, $s_6 = 5$.

Claim 2:

1. The number of 6-shares equals the number of $L5$ shares.

2. There are 30 6-shares, 30 $L5$-shares, and 10 $S5$-shares. (This follows from Part 1.)

Proof of Claim 2:

1) If x is a 6-share then $x \in (\frac{64}{143}, \frac{65}{143})$. hence its buddy is a 6-share in $(\frac{78}{143}, \frac{79}{143})$. Therefore buddy is a bijection of 6-shares to $L5$-shares.

2) There are $s_5 = 3$ 4-students, hence there are 12 5-shares. There are $s_6 = 2$ 5-students, hence there are 10 6-shares. But by part 1 the number of 5-shares and 6-shares is the same. Hence this cannot occur.

End of Proof of Claim 2

Hence we have:

[	$---$	](	$---$	)[	$---$	](	$---$	)[	$---$	]
$\frac{64}{143}$	30 6-shs	$\frac{65}{143}$	0 shs	$\frac{69}{143}$	10 S5 shs	$\frac{74}{143}$	0 shs	$\frac{78}{143}$	30 L5 shs	$\frac{79}{143}$

$\blacksquare$

3 No Protocol for $f(35, 13) \geq \frac{35}{13}$ has a Same Size Shares Student

We prove more than we need in the next theorem. I was hoping it would help later, but it did not.

Theorem 3.1

1. *In a protocol that shows* $f(35, 13) \geq \frac{64}{143}$ *no student gets* ≥ 5 *shares in* $(\frac{64}{143}, \frac{65}{143}]$.

2. *In a protocol that shows* $f(35, 13) \geq \frac{64}{143}$ *every 6-students gets* ≥ 2 *shares of size* $\frac{64}{143}$. *(This follows from Part 1.)*

3. $f(35,13) \le \frac{64}{143}$*. (This follows from Part 2.)*

Proof: Assume, by way of contradiction, that some student gets ≥ 5 shares in $(\frac{64}{143}, \frac{65}{143}]$.

Note: Many of the Claims we need in this proof do not use the premise that that some student gets ≥ 5 *shares in* $(\frac{64}{143}, \frac{65}{143}]$*. When this happens we note it for clarity and sanity.*

These are all 6-shares so it must be a 6-student. So this 6-student uses ≤ 1 share of size $\frac{64}{143}$.

There are 5 more students and they use the remaining 24 6-shares.

Claim 1 and its Proof

1. A 6-student gets ≤ 5 shares of size $\frac{64}{143}$. **Proof:** if he used 6 then he would have $6 \times \frac{64}{143} < \frac{35}{13}$. (Did not use premise.)

2. There are ≤ 21 shares of size $\frac{64}{143}$. **Proof:** By the above and Part 1 (a) one of the 6-students gets ≤ 1 such shares, and (b) the remaining four of the 6-students get ≤ 5 such shares.

3. There are ≤ 21 shares of size $\frac{79}{143}$. **Proof:** These are the buddies of the shares of size $\frac{64}{143}$.

We now look at the 5-people and 5-shares. There are eight 5-people and they will use ≤ 21 shares of size $\frac{79}{143}$, 10 shares in $[\frac{69}{143}, \frac{74}{143}]$, and of course other shares.

Claim 2: and Proof

1. No 5-student gets ≥ 3 S5 shares. **Proof:** $\le 3 \times \frac{74}{143} + 2 \times \frac{79}{143} < \frac{35}{13}$. (Did not use premise.)

2. At least two 5-students get 2 S5 shares and 3 L5 shares. **Proof:** There are eight 5-students. There are 10 S5-shares. By Part 1 no 5-student gets ≥ 3 S5 shares. Hence at least two 5-students get 2 S5 shares. (Did not use premise.)

3. At least two 5-students get two $\frac{74}{143}$'s and three $\frac{79}{143}$'s. **Proof:** The only way a 5-student can get up to $\frac{35}{13}$ using 2 S5 shares and 3 L5 shares is two $\frac{74}{143}$'s and three $\frac{79}{143}$'s. (Did not use Premise)

4. There are at least four $\frac{74}{143}$ shares **Proof:** From last item. (Did not use Premise)

5. There are at least four $\frac{69}{143}$ shares **Proof:** The buddies of the at least four $\frac{74}{143}$. (Did not use Premise)

RECAP:

1. There are eight 5-students. There are ≤ 21 shares of size $\frac{79}{143}$. There are two 5-students that have two $\frac{74}{143}$'s and three $\frac{79}{143}$'s. So, removing those students and shares we have six 5-students and ≤ 15 shares of size $\frac{79}{143}$.

2. There are ≥ 4 shares of size $\frac{69}{143}$ that still have not been used.

3. SO- we have six 5-students who have to use ≤ 15 shares of size $\frac{79}{143}$ and ≥ 4 shares of size $\frac{69}{143}$.

Claim 3 and Proof:

1. No student gets 2 $\frac{69}{143}$'s. **Proof:** $2\frac{69}{143} + 3\frac{79}{143} < \frac{35}{13}$. (Did not use Premise.)

2. At least four 6-students get exactly 1 $\frac{69}{143}$. **Proof:** There are 6 students who have to use 4 $\frac{69}{143}$. From Part 1 nobody can use two of them. (Did not use Premise.)

3. A student who has a $\frac{69}{143}$ has to have four $\frac{79}{143}$. **Proof:** $\frac{69}{143} + 4 \times \frac{79}{143} = \frac{35}{13}$. So need to use all four of the top possible piece. (Did not use Premise.)

4. There are four 5-students who each have 4 $\frac{79}{143}$ shares. **Proof:** From the points above. (Did not use Premise.)

RECAP: Lets inventory the 5-students

Two 5-students use 3 shares of size $\frac{79}{143}$.

Four 5-students use 4 shares of size $\frac{79}{143}$.

Two 5-students we have not looked at.

The 5-students use $2 * 3 + 4 * 4 = 22$ shares of size $\frac{79}{143}$.

Too bad- there are only 21 of them.

Contradiction.

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Theorem 3.2 *In a protocol that shows* $f(35, 13) \geq \frac{64}{143}$ *no student gets same size shares.*

Proof:

There are two cases: a 5-student gets same-size-shares and a 6-student gets same-size-shares.

1) If a 5-student has shares all the same size then they are of size $\frac{35}{5\times 13} = \frac{7}{13} = \frac{77}{143}$. By the diagram above, there are no shares of size $\frac{77}{143}$.

2) If a 6-student has shares all the same size then they are of size $\frac{35}{6\times 13}$. By calculation one can show

$$\frac{64}{143} < \frac{35}{6 \times 13} < \frac{65}{143}$$

Hence there is a student who gets ≥ 5 (in fact, 6) shares in $(\frac{64}{143}, \frac{65}{143}]$. This contradicts Theorem 3.1.

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