CSE 50th Anniversary Celebration at UB

The Classification Program of Counting Problems

Jin-Yi Cai
(University of Wisconsin-Madison)

September 29th, 2017

Turing

Entscheidungsproblem

The rigorous foundation of Computability Theory was established in the 1930s, . . .

Turing, A. M. (1937), “On Computable Numbers, with an Application to the Entscheidungsproblem”, Proceedings of the London Mathematical Society, (Ser. 2, Vol. 42)

Answering a question of Hilbert

Hilbert

Gödel

Church

Von Neumann

Computable Yet Not Efficiently Computable

Given N, how fast can one factor it?

$$N = 5772072129697183320378579117282724 31?$$

Computable Yet Not Efficiently Computable

$$
\begin{aligned}
N' \quad = \quad & 13756295877065550723286378713930120642244218835580062 \\
& 518690227129476541679834062939237944411867525 9?
\end{aligned}
$$

Computable Yet Not Efficiently Computable

$$N' = 137562958770655507232863787139301206422442188355800625186902271294765416798340629392379444118675259?$$

$$N = 9361973132609 \times 616544402332483406165 59$$

Computable Yet Not Efficiently Computable

$$
\begin{aligned}
N' &= 1375629587706555072328637871393012064224421883558006\,2 \\
&\quad 5186902271294765416798340629392379444118675259?
\end{aligned}
$$

$$N = 9361973132609 \times 616544402332483406165 59$$

$$
\begin{aligned}
N' &= 1471865453993855302660887614137521979 \times \\
&\quad 9346163971535797776916355819960689658405123754163818858028032 1
\end{aligned}
$$

Counting Problems

For every problem in NP, one can think of a counting version of the problem.

Counting Problems

For every problem in NP, one can think of a counting version of the problem.

E.g., #SAT: Count the number of satisfying assignments to a Boolean formula.

#VertexCover
#VertexColoring
#EdgeColoring

Counting Problems

For every problem in NP, one can think of a counting version of the problem.

E.g., #SAT: Count the number of satisfying assignments to a Boolean formula.

#VertexCover
#VertexColoring
#EdgeColoring

Valiant introduced the class #P.
Toda's theorem: PH $\subseteq$ $P^{\#P}$.

Ladner's Theorem

Ladner's theorem: If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

Ladner's Theorem

Ladner's theorem: If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

Can be adapted to #P.

Ladner's Theorem

Ladner's theorem: If P $\neq$ NP, then there are problems in NP that are neither in P nor NP-complete.

Can be adapted to $\#$P.

Sum-of-Product computations.

The Classification Program

We aim to achieve complete classifications for various classes of Sum-of-Product computations.

The Classification Program

We aim to achieve complete classifications for various classes of Sum-of-Product computations.

For example we want to give a classification theorem for all counting constraint satisfaction problems (#CSP).

Partition Functions

Coming from statistical physics, the sum-of-product are not merely 0-1 valued products, but have arbitrary weights.

Partition Functions

Coming from statistical physics, the sum-of-product are not merely 0-1 valued products, but have arbitrary weights.

This form is also becoming increasingly used in CS like in AI, Machine Learning . . .

Partition Functions

Coming from statistical physics, the sum-of-product are not merely 0-1 valued products, but have arbitrary weights.

This form is also becoming increasingly used in CS like in AI, Machine Learning . . .

Message passing algorithms.

Dichotomy Theorems

For a broad class of counting problems expressible as sum-of-product computations with arbitrary complex-valued constraint functions, we want to classify every problem within the class to be either solvable in polynomial time, or #P-hard.

Dichotomy Theorems

For a broad class of counting problems expressible as sum-of-product computations with arbitrary complex-valued constraint functions, we want to classify every problem within the class to be either solvable in polynomial time, or #P-hard.

Often we also wish to take into account of the interesting case where a problem is #P-hard in general, but becomes in P when restricted to planar graphs.

Dichotomy Theorems

For a broad class of counting problems expressible as sum-of-product computations with arbitrary complex-valued constraint functions, we want to classify every problem within the class to be either solvable in polynomial time, or #P-hard.

Often we also wish to take into account of the interesting case where a problem is #P-hard in general, but becomes in P when restricted to planar graphs.

Kasteleyn's algorithm, and the power of Valiant's holographic algorithms.

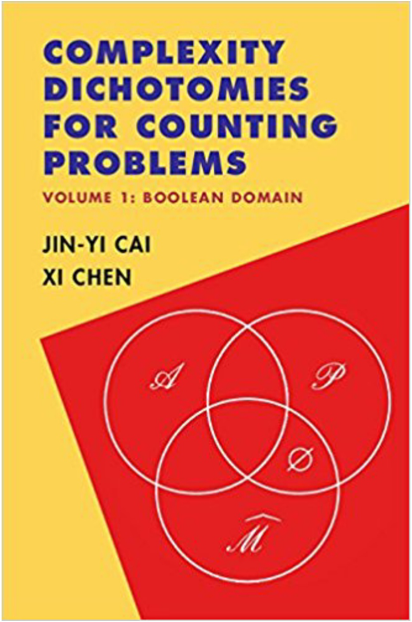
Dichotomy Theorems

For a broad class of counting problems expressible as sum-of-product computations with arbitrary complex-valued constraint functions, we want to classify every problem within the class to be either solvable in polynomial time, or #P-hard.

Often we also wish to take into account of the interesting case where a problem is #P-hard in general, but becomes in P when restricted to planar graphs.

Kasteleyn's algorithm, and the power of Valiant's holographic algorithms.

An overview of the Classification Program.


COMPLEXITY DICHOTOMIES FOR COUNTING PROBLEMS
VOLUME 1: BOOLEAN DOMAIN
JIN-YI CAI
XI CHEN

Three Frameworks for Counting Problems

The following three frameworks are in increasing order of strength.

- Graph Homomorphisms
- Counting Constraint Satisfaction Problems (#CSP)
- Holant Problems

In each framework, there has been remarkable progress in the classification program.

Graph Homomorphisms

Graph Homomorphisms or H-Coloring was defined by Lovász in 1967.

Graph Homomorphisms

Graph Homomorphisms or H-Coloring was defined by Lovász in 1967.

Let

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

be a Triangle.

A graph homomorphism from G to H, is a mapping ξ from $V(G)$ to $V(H)$ such that

$$(u, v) \in E(G) \implies (\xi(u), \xi(v)) \in E(H).$$

I.e., ξ is a THREE-COLORING of G.

Partition Function of Graph Homomorphism

Fix a matrix $A = (A_{i,j}) \in \mathbb{C}^{q \times q}$.
Think of it as defining a binary edge function on an input graph $G = (V, E)$.
Consider all vertex assignments $\xi : V \to [q] = \{1, 2, \cdots, q\}$.
For each $(u, v) \in E$, an assignment ξ gives an evaluation $\prod_{(u,v)\in E} A_{\xi(u),\xi(v)}$. Then the partition function of Graph Homomorphism is

$$Z_A(G) = \sum_{\xi : V \to [q]} \prod_{(u,v)\in E} A_{\xi(u),\xi(v)}.$$

Partition Function of Graph Homomorphism

Fix a matrix $A = (A_{i,j}) \in \mathbb{C}^{q \times q}$.
Think of it as defining a binary edge function on an input graph $G = (V, E)$.
Consider all vertex assignments $\xi : V \to [q] = \{1, 2, \cdots, q\}$.
For each $(u, v) \in E$, an assignment ξ gives an evaluation $\prod_{(u,v) \in E} A_{\xi(u),\xi(v)}$. Then the partition function of Graph Homomorphism is

$$Z_A(G) = \sum_{\xi : V \to [q]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Graph Vertex Coloring

Partition Function of Graph Homomorphism

Fix a matrix $A = (A_{i,j}) \in \mathbb{C}^{q \times q}$.
Think of it as defining a binary edge function on an input graph $G = (V, E)$.
Consider all vertex assignments $\xi : V \to [q] = \{1, 2, \cdots, q\}$.
For each $(u, v) \in E$, an assignment ξ gives an evaluation $\prod_{(u,v) \in E} A_{\xi(u),\xi(v)}$. Then the partition function of Graph Homomorphism is

$$Z_A(G) = \sum_{\xi : V \to [q]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Graph Vertex Coloring

Take binary DISEQUALITY function on each edge.

More Examples

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

then $Z_A(G)$ counts the number of VERTEX COVERS in G.

More Examples

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

then $Z_A(G)$ counts the number of Vertex Covers in G.

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

then $Z_A(G)$ is equivalent to counting the number of induced subgraphs of G with an even number of edges.

Dichotomy Theorems for Graph Homomorphism

- Dyer, Greenhill: $\{0, 1\}$-valued;
- Bulatov, Grohe: Non-negative valued;
- Goldberg, Grohe, Jerrum, Thurley : real-valued;
- C., Pinyan Lu, Xi Chen: complex-valued.

Theorem

There is a complexity dichotomy for $Z_A(\cdot)$:
For any symmetric complex valued matrix $A \in \mathbb{C}^{q \times q}$, the problem of computing $Z_A(G)$, for any input G, is either in P or #P-hard.

The dichotomy criterion is explicit: Given A, whether $Z_A(\cdot)$ is in P or #P-hard can be decided in polynomial time in the size of A.

Dichotomy Theorems for Graph Homomorphism

- Dyer, Greenhill: $\{0,1\}$-valued;
- Bulatov, Grohe: Non-negative valued;
- Goldberg, Grohe, Jerrum, Thurley : real-valued;
- C., Pinyan Lu, Xi Chen: complex-valued.

Theorem

There is a complexity dichotomy for $Z_A(\cdot)$:
For any symmetric complex valued matrix $A \in \mathbb{C}^{q \times q}$, the problem of computing $Z_A(G)$, for any input G, is either in P or #P-hard.

The dichotomy criterion is explicit: Given A, whether $Z_A(\cdot)$ is in P or #P-hard can be decided in polynomial time in the size of A.

SIAM J. Comput. 42(3):924-1029 (2013) [C., Xi Chen, Pinyan Lu] (106 pages)

Counting Constraint Satisfaction Problems (#CSP)

Let $\mathcal{F} = \{f_1, \ldots, f_h\}$ be a finite set of constraint functions:

$$f_i : [q]^{r_i} \to \mathbb{C}$$

Counting Constraint Satisfaction Problems (#CSP)

Let $\mathcal{F} = \{f_1, \ldots, f_h\}$ be a finite set of constraint functions:

$$f_i : [q]^{r_i} \to \mathbb{C}$$

- An instance of $\#\mathrm{CSP}(\mathcal{F})$ consists of variables $x_1, \cdots, x_n$ over $[q]$ and a finite sequence of constraint functions from $\mathcal{F}$, each applied to a sequence of these variables.

Counting Constraint Satisfaction Problems (#CSP)

Let $\mathcal{F} = \{f_1, \ldots, f_h\}$ be a finite set of constraint functions:

$$f_i : [q]^{r_i} \to \mathbb{C}$$

- An instance of $\#\text{CSP}(\mathcal{F})$ consists of variables $x_1, \cdots, x_n$ over $[q]$ and a finite sequence of constraint functions from $\mathcal{F}$, each applied to a sequence of these variables.
  For any assignment $\mathbf{x} = (x_1, \cdots, x_n) \in [q]^n$, let $\mathbf{F}(\mathbf{x})$ be the product of the constraint function evaluations.
- Given an input instance, compute the partition function:

$$\sum_{\mathbf{x} \in [q]^n} \mathbf{F}(\mathbf{x})$$

Dichotomy Theorem for #CSP

- Creignou, Hermann: Boolean domain and $\{0, 1\}$-valued;
- Dyer, Goldberg, Jerrum: Boolean domain and non-negative valued;
- C., Pinyan Lu, Mingji Xia: Boolean domain and complex-valued;
- Bulatov: General domain and $\{0, 1\}$-valued;
- Dyer, Richerby: General domain, $\{0, 1\}$-valued and decidable;
- C., Xi Chen, Pinyan Lu: General domain and Non-negative valued;
- C., Xi Chen: General domain and complex-valued (not known to be decidable).

Dichotomy Theorem for #CSP

- Creignou, Hermann: Boolean domain and $\{0, 1\}$-valued;
- Dyer, Goldberg, Jerrum: Boolean domain and non-negative valued;
- C., Pinyan Lu, Mingji Xia: Boolean domain and complex-valued;
- Bulatov: General domain and $\{0, 1\}$-valued;
- Dyer, Richerby: General domain, $\{0, 1\}$-valued and decidable;
- C., Xi Chen, Pinyan Lu: General domain and Non-negative valued;
- C., Xi Chen: General domain and complex-valued (not known to be decidable).

Theorem

For any domain $[q]$ and any finite set $\mathcal{F}$ of complex-valued constraint functions, #CSP($\mathcal{F}$) is either solvable in polynomial time (if $\mathcal{F}$ satisfies some tractability conditions), or #P-hard (if $\mathcal{F}$ fails these conditions).

J. ACM 64(3): 19:1-19:39 (2017) [C., Xi Chen]

Holant Problems

A signature grid $\Omega = (G, \mathcal{F}, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$, and $f_v : [q]^{\deg(v)} \to \mathbb{C}$.

$$\text{Holant}_\Omega = \sum_{\sigma : E \to [q]} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

where

- $E(v)$ denotes the incident edges of v
- $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$.

Holant Problems

A signature grid $\Omega = (G, \mathcal{F}, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$, and $f_v : [q]^{\deg(v)} \to \mathbb{C}$.

$$\text{Holant}_\Omega = \sum_{\sigma : E \to [q]} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

where

- $E(v)$ denotes the incident edges of v
- $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$.

Here we assume the Boolean domain: $q = 2$.

Holant Problems

A signature grid $\Omega = (G, \mathcal{F}, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$, and $f_v : [q]^{\deg(v)} \to \mathbb{C}$.

$$\text{Holant}_\Omega = \sum_{\sigma: E \to [q]} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

where

- $E(v)$ denotes the incident edges of v
- $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$.

Here we assume the Boolean domain: $q = 2$.

Very natural …, Matching, Edge Coloring, …,
#CSP is a special case.

Dichotomy Theorem for Holant Problems

- Sangxia Huang, Pinyan Lu: Symmetric real-valued functions on Boolean domain;
- C., Heng Guo, Tyson Williams: Symmetric complex-valued functions on Boolean domain.

Dichotomy Theorem for Holant Problems

- Sangxia Huang, Pinyan Lu: Symmetric real-valued functions on Boolean domain;
- C., Heng Guo, Tyson Williams: Symmetric complex-valued functions on Boolean domain.
- C., Pinyan Lu, Mingji Xia: Domain 3, and a single ternary symmetric constraint.
- C., Heng Guo, Tyson Williams: EDGE COLORING, ..., Siegel's Theorem, Galois Theory, ...,

Matching as Holant

$$\text{Holant}_\Omega = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

The problem of counting Perfect Matchings on G corresponds to attaching the Exact-One function at every vertex of G.

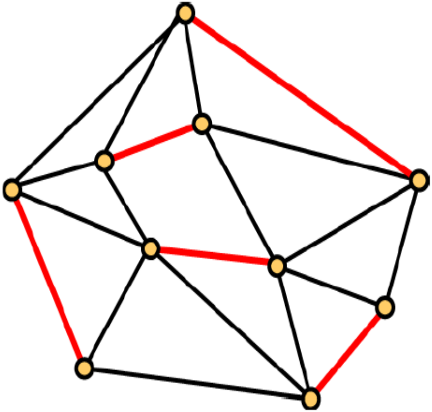
Matching as Holant

$$\text{Holant}_\Omega = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

The problem of counting Perfect Matchings on G corresponds to attaching the Exact-One function at every vertex of G.

The problem of counting all Matchings on G is to attach the At-Most-One function at every vertex of G.

Perfect Matching



Counting Problems with Planar Restriction

If we restrict the input to be planar graphs, new tractable classes arise.

Counting Problems with Planar Restriction

If we restrict the input to be planar graphs, new tractable classes arise.

Fisher, Kasteleyn, and Temperley (FKT) gave a (polynomial-time) algorithm for counting PERFECT MATCHINGS on planar graphs.

Counting Problems with Planar Restriction

If we restrict the input to be planar graphs, new tractable classes arise.

Fisher, Kasteleyn, and Temperley (FKT) gave a (polynomial-time) algorithm for counting PERFECT MATCHINGS on planar graphs.

Pfaffian method.

Counting Problems with Planar Restriction

If we restrict the input to be planar graphs, new tractable classes arise.

Fisher, Kasteleyn, and Temperley (FKT) gave a (polynomial-time) algorithm for counting PERFECT MATCHINGS on planar graphs.

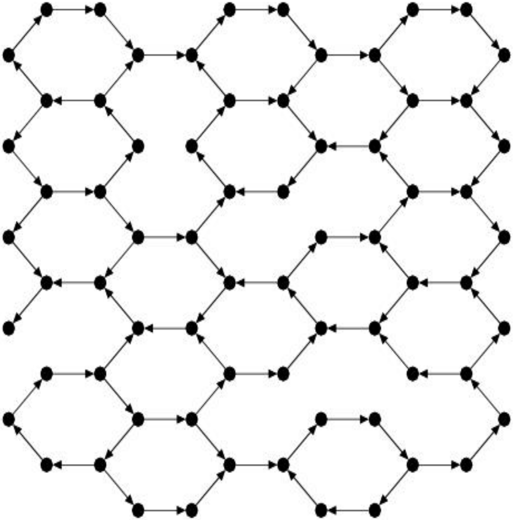
Pfaffian method.

Valiant introduced holographic algorithms which extended the reach of FKT.

Sample Problems Solved by Holographic Algorithms

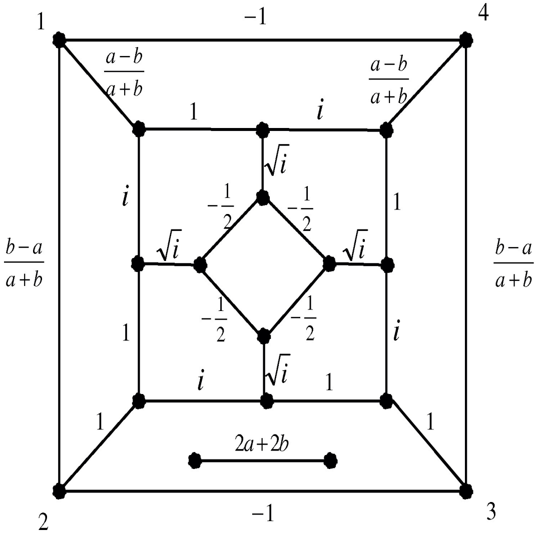
#PL-3-NAE-ICE
Input: A planar graph $G = (V, E)$ of maximum degree 3.
Output: The number of orientations such that no node has all edges directed towards it or all edges directed away from it.

Ising problems are motivated by statistical physics.
Remarkable contributions by Ising, Onsager, Fisher, Temperley, Kasteleyn, C.N.Yang, T.D.Lee, Baxter, Lieb, Wilson etc.



What is the ultimate reach of Valiant's holographic algorithms?

A Matchgate



Classification Theorem

Theorem

For any finite set $\mathcal{F}$ of constraint functions over Boolean variables, each taking complex values and not necessarily symmetric, $\#$CSP($\mathcal{F}$) belongs to exactly one of the three categories according to $\mathcal{F}$:

1. *It is P-time solvable;*
2. *It is P-time solvable over planar graphs but $\#$P-hard over general graphs;*
3. *It is $\#$P-hard over planar graphs.*

Moreover, category (2) consists precisely of those problems that are holographically reducible to the FKT algorithm.

Extended abstract in STOC 2017 [C., Zhiguo Fu].
Full paper at
`https://arxiv.org/abs/1603.07046`.
94 pages.

A Universality Claim

The theorem says:

Holographic algorithms with matchgates are universal for all counting problems in #CSP on Boolean variables that are #P-hard in general but solvable in P over planar structures.

A Universality Claim

The theorem says:

> *Holographic algorithms with matchgates are universal for all counting problems in #CSP on Boolean variables that are #P-hard in general but solvable in P over planar structures.*

This universality is NOT true for the broader class of Holant problems [C., Zhiguo Fu, Heng Guo, Tyson Williams, FOCS 2015].

A Universality Claim

The theorem says:

> *Holographic algorithms with matchgates are universal for all counting problems in #CSP on Boolean variables that are #P-hard in general but solvable in P over planar structures.*

This universality is NOT true for the broader class of Holant problems [C., Zhiguo Fu, Heng Guo, Tyson Williams, FOCS 2015].

. . . even though the FKT algorithm for planar perfect macthings is more properly for a Holant problem.
Thus, that this universality holds for planar #CSP is not self-evident.

A Universality Claim

The theorem says:

> *Holographic algorithms with matchgates are universal for all counting problems in #CSP on Boolean variables <u>that are #P-hard in general but solvable in P over planar structures</u>.*

This universality is NOT true for the broader class of Holant problems [C., Zhiguo Fu, Heng Guo, Tyson Williams, FOCS 2015].

. . . even though the FKT algorithm for planar perfect macthings is more properly for a Holant problem.
Thus, that this universality holds for planar #CSP is not self-evident.

Even without knowing that it is false for planar Holant problems, such a sweeping claim should invite skepticism.

A (Very) Rough Overview of Proof

Much of the proof is carried out in the Holant framework.
Dual perspective:

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}),$$

where $\widehat{\mathcal{C}}$ denotes a holographic transformation by $H_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

A (Very) Rough Overview of Proof

Much of the proof is carried out in the Holant framework.
Dual perspective:

$$\text{Pl-}\#\text{CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}),$$

where $\widehat{\mathcal{C}}$ denotes a holographic transformation by $H_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

We want to show that either $\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which case Pl-#CSP($\mathcal{F}$) is in P, or else Pl-#CSP($\mathcal{F}$) is #P-hard.

A (Very) Rough Overview of Proof

Much of the proof is carried out in the Holant framework.
Dual perspective:

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}),$$

where $\widehat{\mathcal{C}}$ denotes a holographic transformation by $H_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

We want to show that either $\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which case Pl-#CSP($\mathcal{F}$) is in P, or else Pl-#CSP($\mathcal{F}$) is #P-hard.

In the Pl-Holant($\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}$) setting, the tractability condition is expressed as $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{A}}$, or $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$.

A (Very) Rough Overview of Proof

We have $\widehat{\mathscr{A}} = \mathscr{A}$.

A (Very) Rough Overview of Proof

We have $\widehat{\mathscr{A}} = \mathscr{A}$.

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}).$$

$\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$ vs. $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{A}}$, or $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$.

A (Very) Rough Overview of Proof

We have $\widehat{\mathscr{A}} = \mathscr{A}$.

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}).$$

$\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$ vs. $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{A}}$, or $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$.

$\widehat{\mathscr{P}}$ is more difficult to reason about than $\mathscr{P}$, while $\mathscr{M}$ is easier than $\widehat{\mathscr{M}}$ to handle.

A (Very) Rough Overview of Proof

We have $\widehat{\mathscr{A}} = \mathscr{A}$.

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}).$$

$\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$ vs. $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{A}}$, or $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$.

$\widehat{\mathscr{P}}$ is more difficult to reason about than $\mathscr{P}$, while $\mathscr{M}$ is easier than $\widehat{\mathscr{M}}$ to handle.

One necessary condition for $\mathscr{M}$ is the Parity Condition.
Case (1): $\widehat{\mathcal{F}}$ does not satisfy the Parity Condition.
Case (2): $\widehat{\mathcal{F}}$ satisfies the Parity Condition.

A (Very) Rough Overview of Proof

We have $\widehat{\mathscr{A}} = \mathscr{A}$.

$$\text{Pl-\#CSP}(\mathcal{F}) \equiv_T \text{Pl-Holant}(\widehat{\mathcal{EQ}}, \widehat{\mathcal{F}}).$$

$\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$ vs. $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{A}}$, or $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$, or $\widehat{\mathcal{F}} \subseteq \mathscr{M}$.

$\widehat{\mathscr{P}}$ is more difficult to reason about than $\mathscr{P}$, while $\mathscr{M}$ is easier than $\widehat{\mathscr{M}}$ to handle.

One necessary condition for $\mathscr{M}$ is the Parity Condition.
Case (1): $\widehat{\mathcal{F}}$ does not satisfy the Parity Condition.
Case (2): $\widehat{\mathcal{F}}$ satisfies the Parity Condition.

A lucky situation (Proposition 7.12 in paper): If $\widehat{\mathcal{F}}$ satisfies the Parity Condition, then

$$\mathcal{F} \cap \mathscr{P} \subseteq \mathscr{A}.$$

So in this case we do not need to explicitly consider $\widehat{\mathcal{F}} \subseteq \widehat{\mathscr{P}}$.

Some Non-trivial Tractable Problems

Taxicab number 1729, anyone?

Some Non-trivial Tractable Problems

Taxicab number 1729, anyone? Hardy-Ramanujan number.

Some Non-trivial Tractable Problems

Taxicab number 1729, anyone? Hardy-Ramanujan number.

Problem: PI-CrazyPell

Let f be the constraint function on 4 variables:

$$M(f) = \begin{bmatrix} 669669112435114949 & -598015350142588611 & 598015350142588607 & -669669112435114945 \\ 533639108484318913 & -476540387460305851 & 476540387460305855 & -533639108484318909 \\ -533639108484318909 & 476540387460305855 & -476540387460305851 & 533639108484318913 \\ -669669112435114945 & 598015350142588607 & -598015350142588611 & 669669112435114949 \end{bmatrix}$$

Some Non-trivial Tractable Problems

Taxicab number 1729, anyone? Hardy-Ramanujan number.

Problem: PI-CrazyPell

Let f be the constraint function on 4 variables:

$$M(f) = \begin{bmatrix} 669669112435114949 & -598015350142588611 & 598015350142588607 & -669669112435114945 \\ 533639108484318913 & -476540387460305851 & 476540387460305855 & -533639108484318909 \\ -533639108484318909 & 476540387460305855 & -476540387460305851 & 533639108484318913 \\ -669669112435114945 & 598015350142588607 & -598015350142588611 & 669669112435114949 \end{bmatrix}$$

Input : A planar instance of $\#\text{CSP}(f)$.

Output : $$\sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v(\sigma|_{E(v)}).$$

CrazyPell

Let $\widehat{f} = H_2^{\otimes 4} f$, then $\widehat{f}$ has the signature matrix

$$4 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 6437624165826969 8 & 3638760317128320 & 0 \\ 0 & 5694659896305820 80 & 3218812082913484 9 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

One can verify that $\widehat{f} \in \mathscr{M}$. Thus $f \in \widehat{\mathscr{M}}$ and PI-#CSP(f) is tractable.

CrazyPell

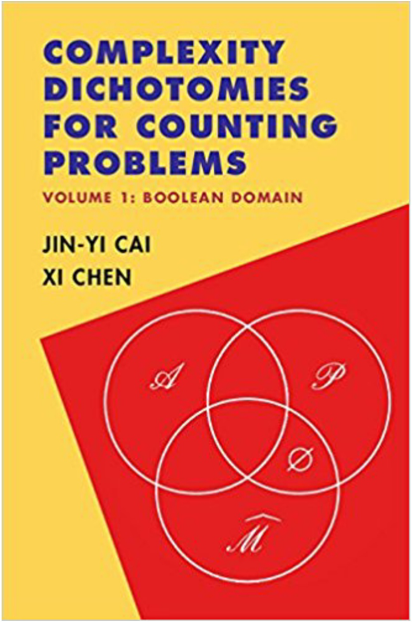
Let $\widehat{f} = H_2^{\otimes 4} f$, then $\widehat{f}$ has the signature matrix

$$4 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 64376241658269698 & 3638760317128320 & 0 \\ 0 & 569465989630582080 & 32188120829134849 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

One can verify that $\widehat{f} \in \mathscr{M}$. Thus $f \in \widehat{\mathscr{M}}$ and PI-#CSP(f) is tractable.

$$(32188120829134849, 1819380158564160)$$

is the smallest integer solution to the Pell's equation $x^2 - 313y^2 = 1$.


COMPLEXITY
DICHOTOMIES
FOR COUNTING
PROBLEMS
VOLUME 1: BOOLEAN DOMAIN
JIN-YI CAI
XI CHEN