Kolkata Algorithms Short Course: III-IV Parallel/Streamable Algorithms and Equation Solving

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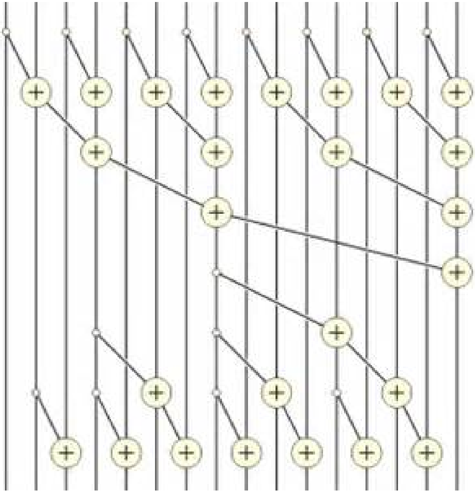
- Another is that sorting has Boolean circuits a power of $\log n$ in *depth*.

Parallel Prefix Sum (PPS): Depth $2 \log n$



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- Answer: use PPS to compose the maps $g_c(q) = \delta(q, c)$ for each character; $g_c \odot g_d =$ take q to $g_d(g_c(q))$ [show on board].

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- Gives Mergesort in $O(n \log n)$ time with $O((\log n)^2)$ depth.

Python code from Wikipedia

```
def bitonic_merge(up, x): # assume input x is bitonic
    if len(x) == 1:
        return x
    else:
        bitonic_compare(up, x)
        first = bitonic_merge(up, x[:len(x) / 2])
        second = bitonic_merge(up, x[len(x) / 2:])
        return first + second

def bitonic_compare(up, x):
    dist = len(x) / 2
    for i in range(dist):
        if (x[i] > x[i+dist]) == up:
            x[i], x[i+dist] = x[i+dist], x[i] #swap
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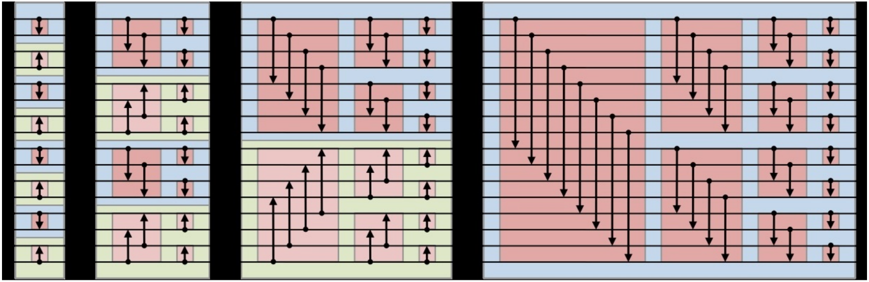
Picture (from Wikipedia)



Theorem: Every decision problem or function in nondeterministc logspace can be processed in parallel by circuits of $n^{O(1)}$ size and $O((\log n)^2)$ depth.

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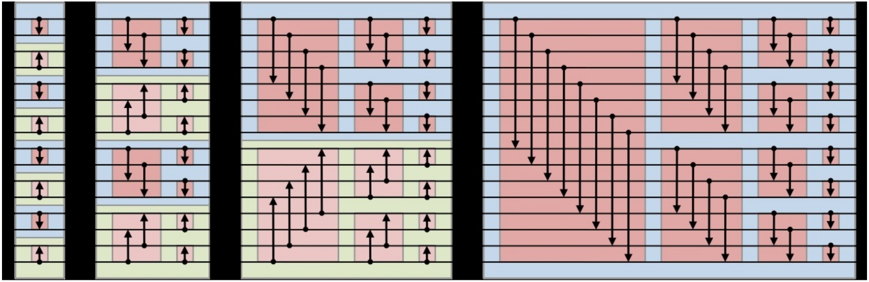

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Thus one reason to care about the theoretical distinction of the "BFS class" is being able to make better parallel/cloud-friendly algorithms.

Solving Arithmetical Equations

A famous example:

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- If the NAND gate has multiple outgoing wires w_i, add equations $w_i = w$.

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- Thus equation solving is **NP-hard**.

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- So 3SAT is in NP and basically so is equation solving—over $\{0,1\}$-solutions anyway.

Definition. A decision problem B is *NP-hard* if for all problems A in NP there is a polynomial-time computable translation function f such that for all inputs x of problem A, the string $y = f(x)$ is an equivalent input of problem B. And B is *NP-complete* if also B is in NP.

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- Each of these variables can appear negated: $\bar{y}_1, \ldots, \bar{y}_m, \bar{u}, \bar{v}, \bar{w}$ etc.
- The key is what we covered in day 2: the memory map of M can be converted into Boolean circuits C_n, one for each n (and the corresponding m) such that M accepts (x, y) if and only if $C_n(x, y) = 1$. We can build C_n using only NAND gates.

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- So 3SAT is NP-hard, and since it is in NP, it is NP-complete. $\square$

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The equations in this proof are indeed *very* simple—degree 2 for the $u_g v_g$ terms and the Boolean equations. Does this really mean that solving them is hard in practice?

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A Standard Greedy Heuristic Algorithm

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set<Clause> TODO = clauses(phi);
set<Variable> FREE = {x_1,...,x_n}
while (TODO and FREE are both nonempty) {
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   Set a_i = true or false accordingly;
   TODO \= {newly satisfied clauses};
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if (empty TODO) {
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   fail; maybe re-try with randomised x_i choices?
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Current “SAT Solvers” use more-sophisticated heuristics.

Equation Solvers Use a Hammer

Represent a given set of pure-arithmetic equations abstractly as

$$\begin{aligned} p_1(z_1, \dots, z_n) &= 0; \\ p_2(z_1, \dots, z_n) &= 0; \\ \vdots &= 0; \\ p_s(z_1, \dots, z_n) &= 0; \end{aligned}$$

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For any polynomials $q_1,\dots,q_s$ in the same variables $\vec{z}$, the polynomial

$$r(\vec{z}) = q_1(\vec{z})p_1(\vec{z}) + q_2(\vec{z})p_2(\vec{z}) + \cdots q_s(\vec{z})p_s(\vec{z})$$

must also be equated to 0. Call it an “algebraic consequence.”

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- Sometimes BA runs for time $\approx 2^{d^n}$ where d is the max degree of the given polynomials $p_1, \ldots, p_s$, which in worst case is double-exponentially horrible.
- But in many cases it finishes quickly enough, so people use it. . .

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