Quantum Computers

And How Does Nature Compute?

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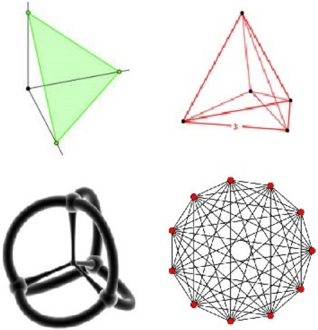
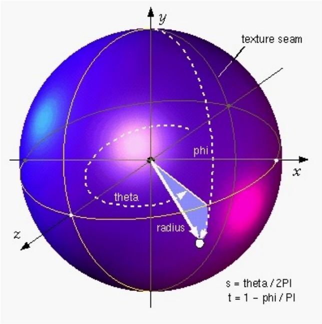
[1] Includes joint work with Amlan Chakrabarti, U. Calcutta

If you were designing Nature, how would you embody probabilities?



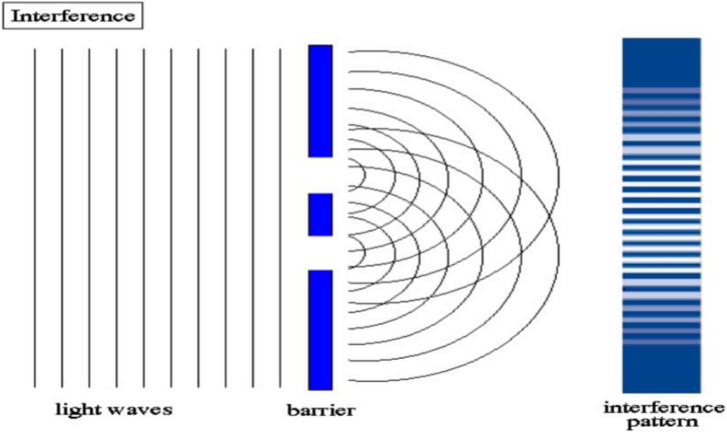


Simplex: $\sum_i p_i = 1$, each $p_i \geq 0$.
Spiky. Understood about 1950.

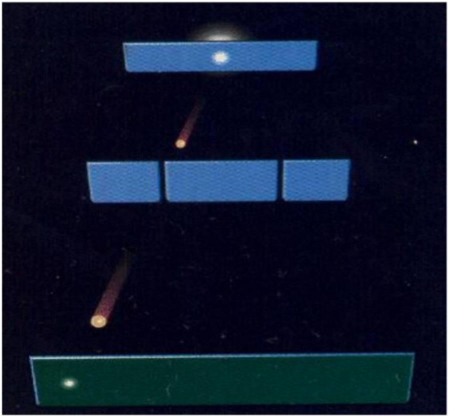
Sphere: $\sum_i |a_i|^2 = 1$; $p_i = |a_i|^2$.
Smooth. Understood by 300 BC.

Amplitudes and the Two-Slit Experiment

The a_i are called **amplitudes** and are physically real quantities.



...which works even when photons go singly!



Nature operates on the a_i. The probabilities p_i are "derivative." But why should Nature have probabilities at all?

Answer(?): She doesn't!

The Schrödinger Equation describes a deterministic process (simplified):

$$U(t) = e^{-iHt/\hbar}.$$

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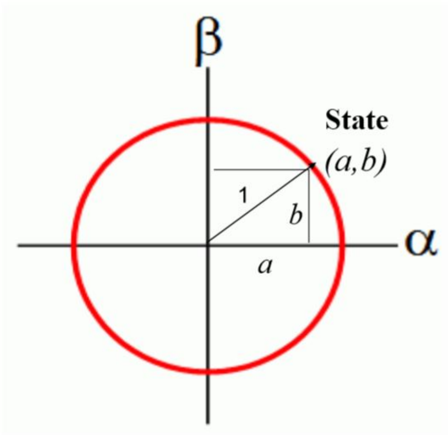
When H has cosmic scale this describes a multi-branch evolution, of which we experience one branch with statistical regularities that we experience as probabilities. When H has tiny scale and $N = 2$ we get a **qubit**.

A Qubit

Quantum Bits, e.g. spins.



Probability of observing Alpha is a-squared, Beta is b-squared. By Pythagoras, these add to 1.

What if we have 17 qubits?

If the qubits are independent, you could represent their state by

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_{17}, b_{17})$$

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You can "precompute" all $2^{17} = 131,072$ combinations by a vector of length $N = 131,072$ defined as the **tensor product** of the little vectors:

$$\vec{a} = (a_1, b_1) \otimes (a_2, b_2) \otimes (a_3, b_3) \otimes \cdots \otimes (a_{17}, b_{17}).$$

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Must *we* do this? Apparently *yes* if we wish to reckon with **entangled** states, which are definable as N-vectors that cannot be decomposed in this way. *Does Nature do this?* That's the \$64,000,000,000 question...

Chalkboard Interlude...

[In the talk I illustrated nondeterministic and deterministic finite automata accepting the languages L_k of binary strings whose k-th from last bit is a 1. The NFA for L_3 needs only 4 states plus a dead state. The minimum DFA for L_3 needs $2^3 = 8$ states, and I drew all its twisted spreading on the board. For $k = 17$ the NFA grows only linearly to 18 states, but the DFA explodes to $2^{17} = 131,072$ states.

Again I posed the question: would we do the DFA or the NFA? What would Nature do? Well I could definitely say what UNIX does with `grep` and Perl and Python similaly when matching length-n lines of text to regular expressions: they build and simulate directly the **NFA**, taking $O(nk)$ time as opposed to $2^k n$ time.

I have not yet fully developed the NFA/DFA analogy to the "wave function of the universe"; reactions thus far are welcome.]

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Using $(1+i)(1-i) = 2$ but $(1+i)(1+i) = 2i$ which cancels $(1-i)(1-i) = -2i$, we get

$$V \cdot V^* = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{but} \quad V \cdot V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

With Two Qubits

For $n = 2$ qubits you need $N = 2^n = 4$ as the vector and matrix dimension. Consider

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The column vector $e_{00} = (1,0,0,0)^T$ stands for the "off-off" state, Then

$$Ue_{00} = \frac{1}{\sqrt{2}}(1,0,0,1)^T = \frac{1}{\sqrt{2}}(e_{00} + e_{11}).$$

This means you have probability $1/2$ of *observing* 00 or 11 as outcomes, but will *never* observe 01 or 10. The two components are entangled.

More Qubits

The $\otimes$ product of vectors is a special case of the $\otimes$ product of matrices:

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,N}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,N}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1}B & a_{N,2}B & \cdots & a_{N,N}B \end{bmatrix}$$

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If we do this n times with the 2×2 **Hadamard matrix**

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then we get the **Hadamard transform** $H_N = H^{\otimes n}$. On argument $e_{00\ldots0}$ it produces the maximally superposed state

$$\frac{1}{\sqrt{2^n}}(1, 1, 1, \ldots, 1) = \frac{1}{\sqrt{2}}(1, 1) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(1, 1).$$

Quantum Fourier Transform

With $\omega = e^{2\pi i/N}$, the ordinary Fourier matrix F_N is:

$$\frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{N-2} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{N-3} \\ \vdots & & & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{N-2} & \omega^{N-3} & \cdots & \omega \end{bmatrix}$$

That is, $F_N[i,j] = \omega^{ij \bmod N}$. As a "piece of code," it's simple.

What's "quantum" is the assertion that Nature provides sufficiently close approximations to this with about order-n^2 effort when $N = 2^n$. (Note also $F_N e_{00\cdots 0} = H_N e_{00\cdots 0}$.)

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- Factoring numbers M allows breaking the RSA cryptosystem with effort roughly $O(n^3)$, whereas the best known on classical computers is roughly $2^{n^{1/3}}$.

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- I moderated a debate on the "Gödel's Lost Letter" blog between Gil Kalai and Aram Harrow, all during 2012. Richard Lipton and I are beginning to update it for a book.
- At the heart are schemes for **quantum error-correcting codes**, also partly originated by Shor, and the Quantum Fault Tolerance Theorem giving an absolute physical threshold which if met by the raw **decoherence** error rate enables the codes to succeed.

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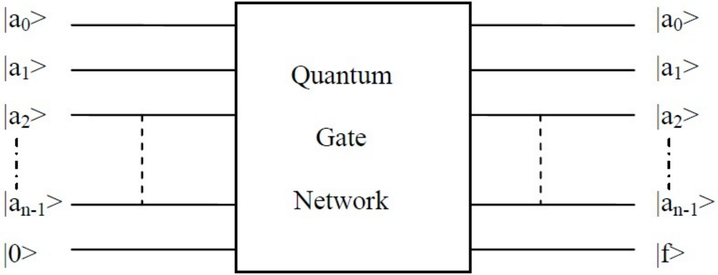
- To conclude: factoring and breaking RSA follow **if we can find *human* notation for how *Nature* "really" computes.**
- My own research tries to find Nature's secret in the algebra of multi-variable polynomials, into which **quantum circuits** can be translated. A more-technical version of the talk would include the following slides on quantum circuits, then show my blog article rjlipton.wordpress.com/2012/07/08/grilling-quantum-circuits/ .]

Quantum Circuits

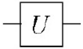
Quantum circuits look more constrained than Boolean circuits:



But Boolean circuits look similar if we do Savage's TM-to-circuit simulation and call each *column* for each tape cell a "cue-bit."

Quantum gates

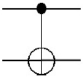
single qubit operation:



controlled-NOT:

control

target

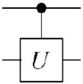


unitary matrix $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

controlled-U:

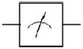
control

target



unitary matrix $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$

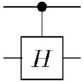
measurement in the $|0\rangle, |1\rangle$ basis:



[Slides concept by D. Bacon, U Washington]

Quantum gates: an example

controlled-gate
(here controlled-H)



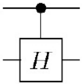
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

input: $|\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$

output: $|\psi'\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

compute:

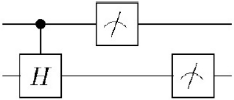
$|1\rangle$
$|1\rangle$



$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

measure:

$|1\rangle$
$|1\rangle$



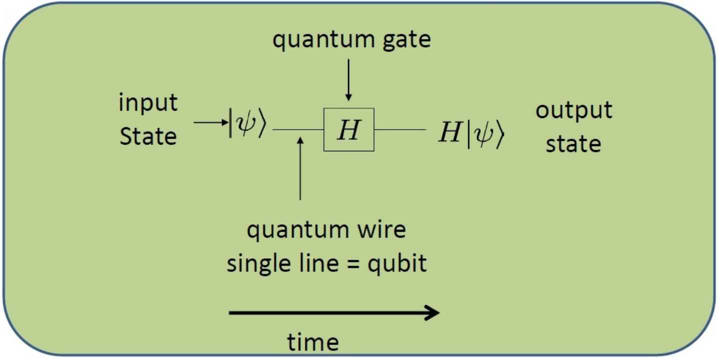
Probability of 10: $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

Probability of 11: $\left|\frac{-1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

Probability of 00 and 01: $|0|^2 = 0$

Quantum circuits

Quantum circuit diagrams to visualize a computation:



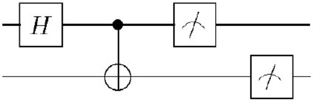
Quantum circuits are sequences of instructions. Describes a series of unitary evolutions (quantum gates) applied to a quantum state.

Quantum circuit example

$$H \otimes \mathbf{1}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \mathbf{1}_2$$

$|0\rangle$

$|0\rangle$



Prob(meas. $|00\rangle) = 0.5$

Prob(meas. $|11\rangle) = 0.5$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

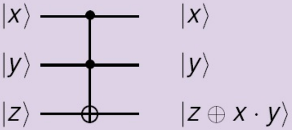
$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |11\rangle \\ |11\rangle &\to |10\rangle \end{aligned}$$

Toffoli Gate

The Toffoli gate "TOF"

x	y	z	x'	y'	z'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0



Theorem (Toffoli, 1981)

Any reversible computation can be realized by using TOF gates and ancilla (auxiliary) bits which are initialized to 0.

Slides by Martin Rötteler

Bounded-error Quantum Poly-Time

A language A belongs to BQP if there are uniform poly-size quantum circuits C_n with n data qubits, plus some number $\alpha \geq 1$ of "ancilla qubits," such that for all n and $x \in \{0, 1\}^n$,

$$\begin{aligned} x \in A &\implies \Pr[C_n \text{ given } \langle x0^\alpha| \text{ measures 1 on line } n+1] > 2/3; \\ x \notin A &\implies \Pr[\ldots] < 1/3. \end{aligned}$$

One can pretend $\alpha = 0$ and/or measure line 1 instead. One can also represent the output as the "triple product" $\langle a \mid C \mid b \rangle$, with $a = x0^\alpha$, $b = 0^{n+\alpha}$.

Two major theorems about BQP are:

(a) C_n can be composed of just Hadamard and Toffoli gates [Y. Shi].

(b) Factoring is in BQP [P. Shor].

[Segue to "Grilling Quantum Circuits" post on GLL blog.]