Reading, Analyzing, and Simulating Quantum Circuits

(With speculation on the status of "quantum supremacy")

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[1]Includes joint work with Amlan Chakrabarti, University of Calcutta, and Chaowen Guan, University of Cincinnati

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  - We know $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE}$ but have not proved $\mathsf{P} \neq \mathsf{PSPACE}$.
- So Math can outpace its own calculations, but can Nature?

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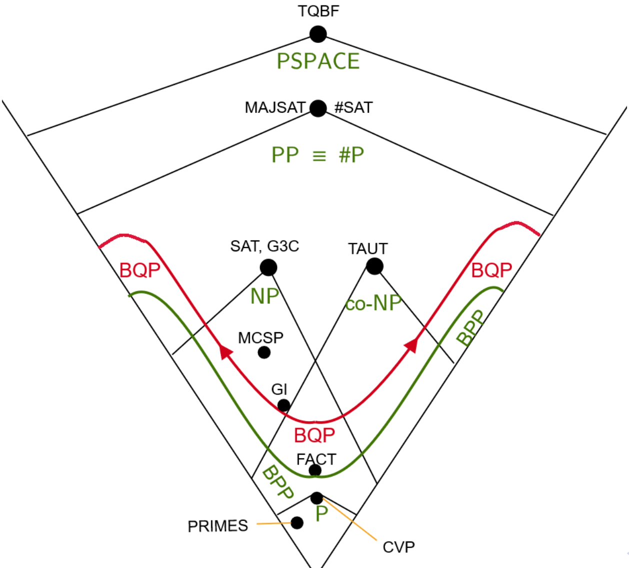
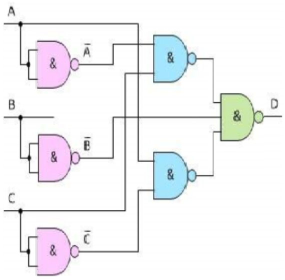
- Hence a shock in 1993-94 when Peter Shor put factoring into **BQP**.
- Realizable by **quantum circuits** of size $\tilde{O}(m^2)$.

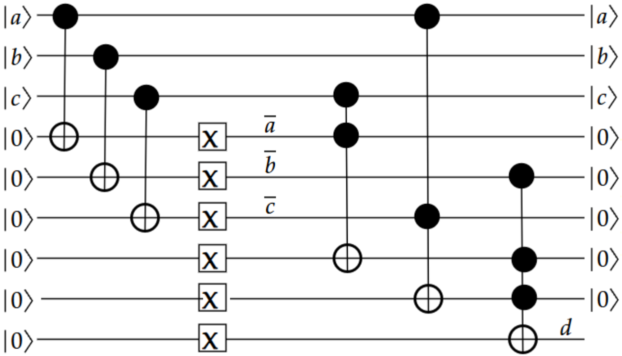
The Complexity Class Neighborhood



Classical and Quantum Circuits

$d = f(a,b,c) = \bar{\wedge}\,(c \;\bar{\wedge}\; \bar{a}, \bar{b}, a \;\bar{\wedge}\; \bar{c})$ $\quad$ $\mathbf{X}$ = NOT, ●—⊕ = controlled-NOT





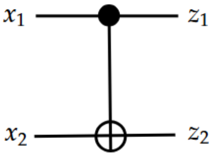
Quantum circuit computes the **reversible** Boolean function
$F(a,b,c,e_1,e_2,e_3,e_4,e_5,e_6) = (a,b,c,e_1,e_2,e_3,e_4,e_5,e_6 \oplus f(a,b,c))$.
Underlying: a vector of $N = 2^9 = 512$ dimensions.

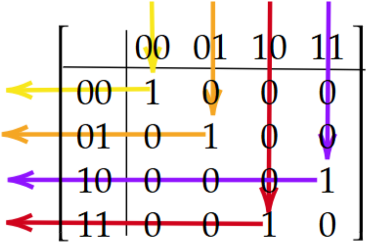
Quantum Coordinates and the CNOT Gate

By **linearity**, an n-qubit circuit is determined by its actions on the 0-1 **standard basis** vectors $e_{0^n} = [1,0,0,\dots,0]^T$ thru $e_{1^n} = [0,0,0,\dots,1]^T$.

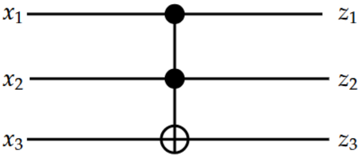
For $n=2$, $e_{00} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $e_{01} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $e_{10} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$; and $e_{11} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ (in "big-endian" order). Then **CNOT**$e_{00} = e_{00}$ and **CNOT**$e_{01} = e_{01}$, while **CNOT**$e_{10} = e_{11}$ and **CNOT**$e_{11} = e_{10}$. **Feynman path** visualization:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$





Toffoli Gate ($n = 3,\ N = 2^n = 8$)



$$
\left[\begin{array}{c|cccccccc}
 & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\hline
000 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
001 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
010 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
011 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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110 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
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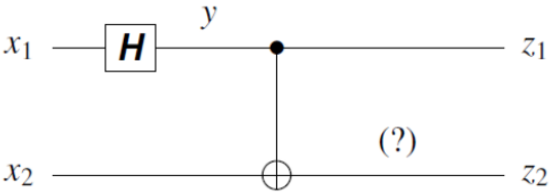
\end{array}\right]
$$

In **Dirac notation**, e_x is written $|x\rangle$. So we have **Tof** $|000\rangle = |000\rangle$ thru **Tof** $|101\rangle = |101\rangle$, while **Tof** $|110\rangle = |111\rangle$ and **Tof** $|111\rangle = |110\rangle$.

Note that fixing $x_3 = 1$ makes $z_3 = x_1 \barwedge x_2$. Since NAND is a universal gate, this already suffices to show that quantum circuits simulate classical ones. Their extra power comes from one more gate.

Hadamard Gate, Nondeterminism, and Entanglement

Hadamard gate H $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ "emits" a free Boolean variable y.



On input $|00\rangle$, $y = 0$ leads to output $z_1 = 0$, $z_2 = 0$, while $y = 1$ leads to $z_1 z_2 = 11$. The other combinations 01 and 10 cannot happen. *Matrices:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

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- **Measuring** the state vector $|\phi\rangle = [a_0, a_1, \ldots, a_{N-1}]^T$ of all n qubits returns some i, $0 \leq i \leq N-1$, with probability $|a_i|^2$. You can say it returns e_i, $|i\rangle$, or the i-th binary string $z_i \in \{0,1\}^n$.

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- This entails that $|\phi\rangle$ is a Euclidean **unit vector**, so that the probabilities sum to 1.
- **Unitary** matrices A, meaning $AA^* = I$, preserve unit vectors.
- Measuring the **entangled** state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ returns $|00\rangle$ with probability 0.5 or $|11\rangle$ with probability 0.5, but never $|01\rangle$ or $|10\rangle$.

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- **Shor's algorithm** creates $D_{C,x}$ such that sampling often gives z from which the period r of $f_a(u) = a^u \bmod x$ can be classically inferred, which in turn often enables factoring x.

Some More Gates

$$\mathsf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

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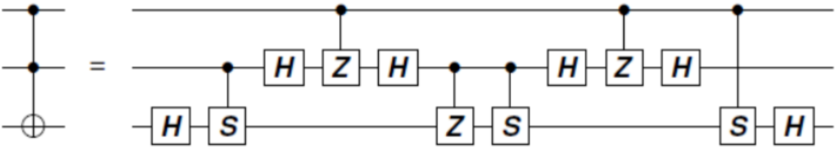
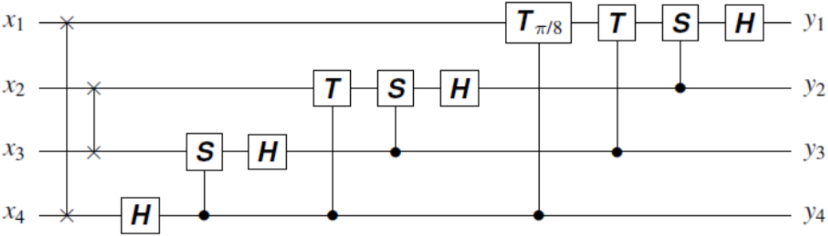
- Note: $\mathsf{T}^2 = \mathsf{S}$, $\mathsf{S}^2 = \mathsf{Z}$, $\mathsf{Z}^2 = \mathsf{I} = \mathsf{H}^2$, and $\mathsf{CS}^2 = \mathsf{CZ}$.

Two Notable Circuits

H and **CS** alone can simulate the Toffoli gate:

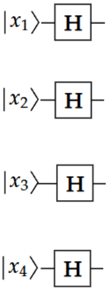


The 4-qubit **Quantum Fourier Transform** == the 16×16 **Discrete Fourier Transform**. In general, $\mathbf{QFT}_n$ needs $O(n^2)$ basic gates.



Juxtaposition and Tensor Product

Many QCs begin with m Hadamard gates on each of m qubits



$$\frac{1}{4}\left[\begin{array}{c|cccccccccccccccc}
\mathbf{H}^{\otimes 4} & 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
\hline
0000 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0001 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
0010 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
0011 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
0100 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
0101 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
0110 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
0111 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1000 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1001 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1010 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
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1100 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
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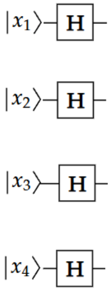

$\frac{1}{4}$

$\mathbf{H}^{\otimes 4}$	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
0000	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0001	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
0010	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
0011	1	−1	−1	1	1	−1	1	−1	1	−1	−1	1	1	−1	1	−1
0100	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
0101	1	−1	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1
0110	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	1	1
0111	1	−1	−1	1	−1	1	1	−1	1	−1	−1	1	−1	1	1	−1
1000	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
1001	1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
1010	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1
1011	1	−1	−1	1	1	−1	1	−1	−1	1	1	−1	−1	1	1	−1
1100	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
1101	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	1	−1	1	−1
1110	1	1	−1	−1	−1	−1	1	1	−1	−1	1	1	1	1	−1	−1
1111	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	−1	−1	1

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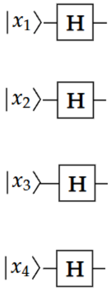

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1001	1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1
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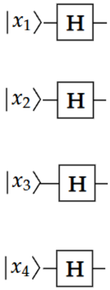

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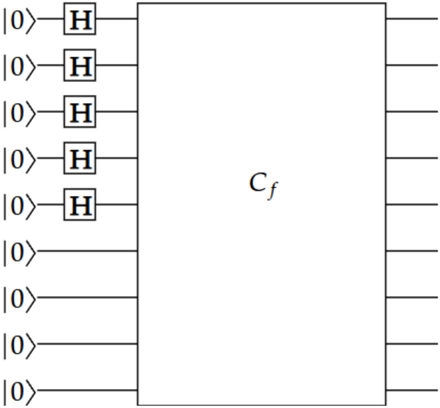
- On the other, some claim this involves splitting off 2^n branches of a multiverse.

Hadamard Transforms and Functions

Suppose $C_f(x, y) = (x, y \oplus f(x))$ computes the reversible form of f. Then



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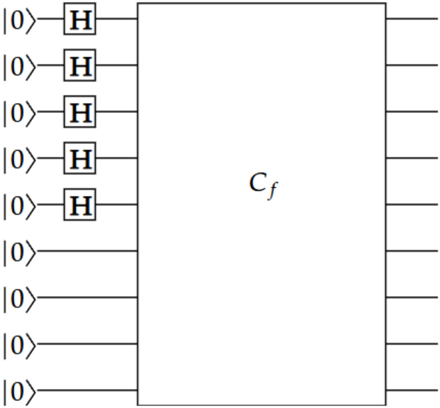

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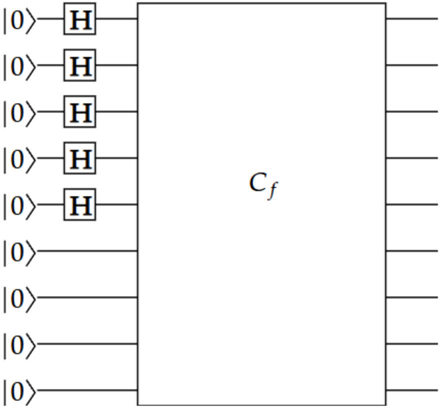

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- *We've now seen all the ingredients of Shor's Algorithm.*

Three Universal Libraries, Phase Angles, and Noise

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5. Iteratively program **tensor network and SVD approximations**... **On the agenda...**

I. Feynman Path Polynomials and Logical Formulas

Let C have "minphase" $K = 2^k$ and let F embed K-th roots of unity ω.

- $\mathsf{H} + \mathsf{Tof}$ has $k = 1$, $K = 2$.
- $\mathsf{H} + \mathsf{CS}$ has $k = 2$, $K = 4$.
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Any QC C of n qubits quickly transforms into a polynomial $P_C = \prod_g P_g$ over gates g and a constant $R > 0$ such that for all $x, z \in \{0,1\}^n$:

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- T-gate also goes cubic.

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Theorem (C. Guan in RCG 2018)

Given C, n, K, h as above, we can quickly build a Boolean formula ϕ_C in variables $y_1, \ldots, y_h$, together with substituted-for $x_1, \ldots, x_n$, $z_1, \ldots, z_n$, and other "forced" variables such that for all $x, z \in \{0,1\}^n$:

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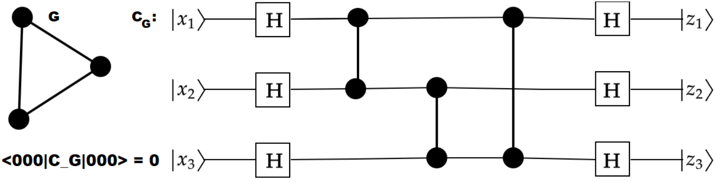
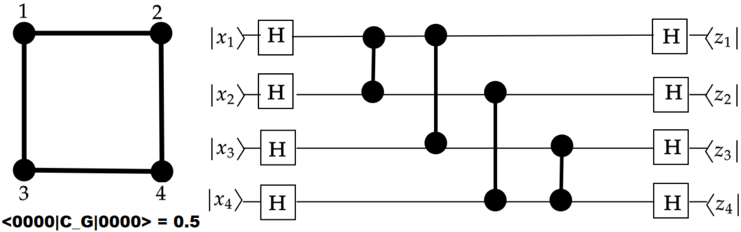
- For $K = 2, 4$ (i.e., for $\mathsf{H} + \mathsf{Tof}$ and $\mathsf{H} + \mathsf{CS}$), we get the acceptance *probability* as a simple difference:

$$|\langle z \mid C \mid x \rangle|^2 = \frac{1}{R} \left(\#sat(\phi_C) - \#sat(\phi'_C) \right).$$

II. Strong Simulation of Graph State Circuits

Computing amplitudes $\langle z \mid C \mid x \rangle$ for Clifford circuits C can be efficiently reduced to computing $\langle 0^n \mid C_G \mid 0^n \rangle$ for **graph-state circuits** C_G of graphs G, using H and CZ gates, as exemplified by:





Improved From $O(n^3)$ to $O(n^{2.37155...})$

Theorem (Guan-Regan, 2019)

For n-qubit stabilizer circuits of size s, $\langle z \mid C \mid x\rangle$ can be computed in $O(s + n^{\omega})$ time, where $\omega \leq 2.37155...$ is the exponent of multiplying $n \times n$ matrices.

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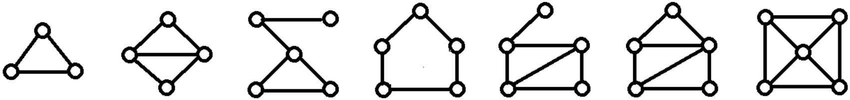

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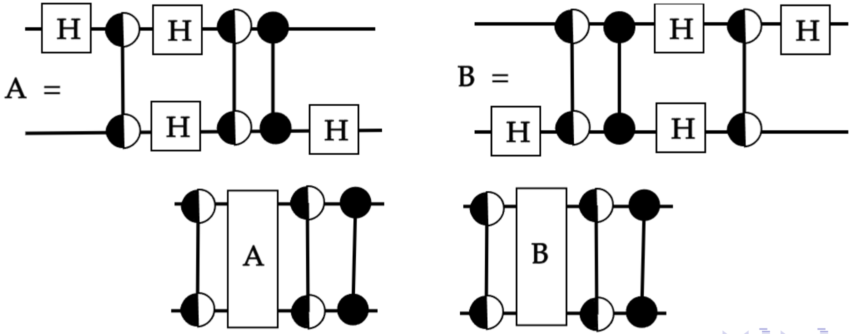

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- We took them in a different direction.

Singular Value Decomposition

Unlike with diagonalization, this is *always* possible:

SVD Theorem: For every $m \times n$ matrix A we can efficiently find:

- an $m \times m$ unitary matrix U,
- an $m \times n$ pseudo-diagonal matrix Σ with non-negative entries $\Sigma[i,i] = \sigma_i$, and
- an $n \times n$ unitary matrix V,

such that $A = U\Sigma V^*$. Furthermore, we can arrange that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$, and in consequence:

- $||A||_F = \sqrt{\sum_i \sigma_i^2}$,
- $||A||_2 = \sigma_1$,
- $A^*A = V\Sigma^T U^* U \Sigma V^* = V \operatorname{diag}\left(\sigma_i^2\right) V^*$, and
- $AA^* = U\Sigma V^* V \Sigma^T U^* = U \operatorname{diag}\left(\sigma_i^2\right) U^*$,

so that the squares of the σ_i and associated vectors give the spectral decompositions of the Hermitian PSD matrices A^*A and AA^*, respectively.

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- Most workable scheme?
- How this might be programmed.

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"It From Bit" we once proclaimed,
but now the Bit has bit the dust
of whizzing quantum chips that gamed
coherence, to evade the trust
that the Word framed creation's hour:
Mother Nature fully lexical.
Why not evolve us that same power?
It is a status most perplexical.