# CSE191 Assignment 6 Answer Key Fall 2013

## Problem 1, Page 126, Exercise 2.1.26

Here is the rhythm of a typical "subset" proof: Let any element of $A \times B$ be given. Show that element belongs to $C \times D$. Since the element is arbitrary, universal generalization gives you that $A \times B \subseteq C \times D$. By definition of Cartesian Product, the element has the form $(a, b)$ where $a \in A$ and $b \in B$. By $A \subseteq C$, we have $a \in C$. Similarly, $b \in D$. Again by definition of Cartesian Product, we have $(a, b) \in C \times D$. Therefore $A \times B \subseteq C \times D$

## Problem 2, Page 126, Exercise 2.1.40

$(A \times B) \times (C \times D) = \{((a, b), (c, d)) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\}$
$A \times (B \times C) \times D = \{((a, (b, c), d)) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\}$
The elements of $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ have different structures.
Therefore, they are not the same.

However, we can give a natural 1–1 correspondence between these sets—even better, we can give 1–1 correspondences between both and the simpler set $\{(a, b, c, d) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\}$.
Namely, define:
$f_1 : \{((a, b), (c, d)) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\} \rightarrow \{(a, b, c, d) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\}$,
that is $f_1(((a, b), (c, d))) = (a, b, c, d)$;
$f_2 : \{((a, (b, c), d)) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\} \rightarrow \{(a, b, c, d) : a \in A \wedge b \in b \wedge c \in C \wedge d \in D\}$,
that is $f_2((a, (b, c), d)) = (a, b, c, d)$.
These functions are clearly onto $A \times B \times C \times D$, and are 1–1 because e.g. if $f_2((a, (b, c), d)) = f_2((a', (b', c'), d')) = (a, b, c, d)$, then we must have $a' = a$, $b' = b$, $c' = c$, and $d' = d$ separately, whicgh makes the arguments $((a, (b, c), d))$ and $((a', (b', c'), d'))$ identical. In this sense we can say that the two ways of "nesting up" a 4-tuple give essentially the same data content.

## Problem 3, Page 137, Exercise 2.2.42

The "zippy" proof hinted by how the question was presented is to realize that $\oplus$ of sets corresponds to bitwise XOR of the bit-vector representations of the sets (as covered in lecture), which in turn is the same as adding their entries mod 2. Since the "mod" step can be left to the end, this addition obeys the same rules as ordinary addition, and hence is associative and commutative. Hence the expressions—including the "flip-a-coin" alternative, all evaluate to the same result.

A direct proof didn't have to be painfully long—here is one:
$A \oplus B = A \cup B - A \cap B = B \oplus A$
$(A \oplus B) \oplus C$
$= (A \cup B - A \cap B) \oplus C$
$= A \cup B \cup C - A \cup B \cap C - A \cap B \cup C + A \cap B \cap C$
$= A \oplus (B \oplus C)$ This proves the commutative and associative laws for $\oplus$. Therefore,
$(A \oplus B) \oplus (C \oplus D)$
$= ((A \oplus B) \oplus C) \oplus D$
$= (A \oplus (B \oplus C)) \oplus D$
$= (A \oplus C \oplus B) \oplus D$
$= (A \oplus C) \oplus (B \oplus D)$
Similarly,
$(A \oplus B) \oplus (C \oplus D) = A \oplus (C \oplus (B \oplus D))$

## Problem 4, Page 137, Exercise 2.2.50

For any $x$, $x \in \bigcup_i A_i$ means $(\exists i)\ x \in A_i$, and $x \in \bigcap_i A_i$ means $(\forall i)\ x \in A_i$. With this in mind:
c) With $A_i = (0, i)$, for all real numbers $x > 0$, there exists an $i \in \mathbb{Z}^+$ such that $x \in A_i$, so $\bigcup_{i=1}^{\infty} A_i = (0, \infty) = \mathbb{R}^+$. But if $x$ is such that for all $i \in \mathbb{Z}^+$, $0 < x$ and $x < i$, then we must have $0 < x < 1$. And vice-versa), so $\bigcap_{i=1}^{\infty} A_i = (0, 1)$.

d) With $A_i = (i, \infty)$, for all real numbers $x > 1$, there exists an $i \in \mathbb{Z}^+$ such that $x \in A_i$, namely with $i = 1$ since $A - 1$ is a superset of all the others. So $\bigcup_{i=1}^{n} A_i = A_1 = (1, \infty)$. But $\bigcap_{i=1}^{\infty} A_i = \emptyset$ because given any $x$, it cannot be $> i$ for all $i$.

## Problem 5, Page 153, Exercise 2.3.22

b) $f(x) = -3x^2 + 7$ is neither one to one nor onto.
$\because f(x) \leq 7, \forall x \in \mathbb{R}$
$\therefore f(x)$ is not onto.
if $y = f(x) < 7$
$x = \pm\sqrt{\frac{7-y}{3}}$ So $f(x)$ is not one to one.
Let's say $x \in [0, \infty), y \in (-\infty, 7]$ Then $y = f(x)$ is bijection, and $x = \sqrt{\frac{7-y}{3}}$

c) $f(x) = \frac{x+1}{x+2}$ is not a function because there is no definition when $x = -2$.
By excluding the bad point $x = -2$, we get a function. But it's still not onto.
$y = f(x) = \frac{x+1}{x+2} = 1 - \frac{1}{x+2} \neq 1$ As a result, we also exclude $y = 1$.
Therefore,
$f : (-\infty, -2) \bigcup (-2, +\infty) \to (-\infty, 1) \bigcup (1, +\infty)$,
$y = f(x) = \frac{x+1}{x+2}$ is a bijection, and $x = \frac{1}{1-y} - 2$

## Problem 6, Page 168, Exercise 2.4.12

a) $\because 0 = -3 \cdot 0 + 4 \cdot 0$,
$\therefore a_n = 0$ is a solution.

b) $\because -3 \cdot 1 + 4 \cdot 1 = 1$
$\therefore a_n = 1$ is a solution.

c) $(-4)^n = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2}$
Divided $(-4)^{n-2}$ out from both sides, we have,
$16 = -3 \cdot (-4) + 4 = 16$
therefore, $a_n = (-4)^n$ is a solution.

## Problem 7, Page 169, Exercise 2.4.32

b) $\sum_{j=0}^{8}(3^j - 2^j) = \sum_{j=0}^{8} 3^j - \sum_{j=0}^{8} 2^j = \frac{1-3^9}{1-3} - \frac{1-2^9}{1-2} = 9330$.

d) $\sum_{j=0}^{8}(2^{j+1} - 2^j) = \sum_{j=0}^{8} 2^j = \frac{1-2^9}{1-2} = 511$.