A heap is a full binary tree, minus some subset of leaves from right-to-left, whose comparable keys obey the invariant that if \( u \) is a child of \( v \), then \( u.key \leq v.key \).

This defines a max-heap and means that root.key is always the largest value in the heap (not necessarily uniquely). There is a corresponding notion of min-heap where the root holds the least value and children compare \( \geq \) their parents. Unlike in a BST:

(a) the left-to-right layout does not matter to the heap's operation or meaning;
(b) the up/down sortedness is too weak to help you find a key in less than linear time.

By (a), we may as well suppose the tree is laid out so that the leaves in the bottom level fill in left-to-right. The text uses the standard term complete tree for this, but I prefer "raster tree" to connote that the tree fills in level-by-level going down and left-to-right in each level.

The signature implementation of a heap is an array with \( n \) elements. Happily, the text defines the version that does not use cell 0 of the array. In ancient times, there was no special support for the swap operation, so it was further convenient to use cell 0 as swap space. Then:

- Nodes \( u, v, \ldots \) are identified with indices in \([1 \ldots n]\), \( n \) treated inclusively in an array of size \( n + 1 \), and 1 being the root;
- The parent of a non-root node \( v \) is always \( u = \lfloor v/2 \rfloor \) (integer division).
- The left child of a non-leaf node \( u \) has index \( 2u \), while the right-child (if present) has \( 2u + 1 \).
- To facilitate adding and removing elements---and optimize memory usage generally---the array has a fixed size \( c \) (so \( n \) can be up to \( c - 1 \)) with the current \( n + 1 \) maintained as a field or variable end apart from the array.

Example (different from the text's)
• Insert 86: it "bubbles up" to the position currently occupied by 84..
• Then pop 97. The right-hand 55 key which got swapped into the last position is first lifted to the root. Then it "sinks down" (like a stone, says the text), always swapping with the larger child, so that 91 is the new max and it settles where 59 was.

Note that keys can jump around after a pop. But the time is never worse than proportional to the height of the tree, which by its enforced balance is $O(\log n)$.

If we add 5 more elements, we will make a full binary tree and (since $c = 32$) also fill the capacity of the array. If we add one more element, we will have to double the size of the array and recopy the elements. The ethic with heaps, in my (ancient) experience, is to right-size them to begin with, because the idea is that they use memory in-place without using temporary storage for copying (like QuickSort versus MergeSort). For like reasons, the code could be sped up [Important note: this meaning of heap is not related to that of the "system heap", whose ethic is to be expandable without the layout regularity needed for values on the "system stack" (which really is organized like a stack)].

**Coding Heaps**

This time I'll use several more helper functions than the text does. Scala seems not to have low-level support for fast swap like C/C++ implementations do, so we'll use heap(0) like the text does. (What follows is not real code inside a Heap or PriorityQueue class.)
def swap(heap, i, j): Unit = {     //REQ: 1 <= i, j <= n < end
    heap(0) = heap(i)
    heap(i) = heap(j)
    heap(j) = heap(0)
}

Some texts define left(u) = 2u and right(u) = 2u+1 and parent(u) = u/2, but this does not much matter. Here are the two most important helpers, not used as-such in the text.

def fixDown(u): Unit = {
    var st = u  //"stone" in text. Note var needed in Scala for val params
    while (st < end/2) {
        val gc = if (keyComp(heap(2*st),heap(2*st+1)) > 0 then 2*st else 2*st+1
        if (keyComp(heap(st), heap(gc)) < 0) {
            swap(st,gc)
            st = gc
        } else {
            return
        }
    }
}

def fixUp(u): Unit
    var bubble = u  //note bubble > = 2 means parent exists and is bubble/2
    while (bubble >= 2 && keyComp(heap(bubble),heap(bubble/2)) > 0) {
        swap(bubble,bubble/2)
        bubble = bubble / 2  //integer division
    }
}

Side note: If one could store a "plus infinity" value in cell 0 as a sentinel value, then one could marginally simplify and speed the while loop by removing the bubble >= 2 check. Since 1 / 2 = 0 with integer division, the loop would stop at the root anyway. But modern branched execution mostly negates the supposed gain, which was dubious even in ancient times.

Now we can quickly code the main operations. Because both fixDown and fixUp run in $O(\log n)$ time, so do these operations.

def enqueue(item): Unit = {
    if (end >= capacity) { ... }  //blah-blah...
    heap(end) = item
    fixUp(end)
end += 1
}

def pop(): A = { //dequeue in text and Scala generally
    val ret = heap(1)
    swap(1,end-1)       //sometimes it is useful to put max there
    fixDown(1)
    end -= 1
    return ret
}

Making and Sorting Heaps (not all in the text)

To create a heap from an array \( arr \) of items in any order, one can create an empty heap and enqueue the items one by one. That would take \( O(n \log n) \) time on the whole, using repeated calls to fixUp. There is a quicker way to make a heap directly from \( arr \), assuming there is at least one unused cell \( n \) at the end. (Well, not having to make this assumption is the main reason other authorities define heaps with the root in cell 0.)

def makeHeap(arr: Array[A], n: Int): Unit = {  //REQ: arr(n) exists and is empty
    arr(n) = arr(0)
    end = n+1
    for (u <- end/2 to 1 by -1) {  //INV: Sub-heaps rooted at v > u are valid
        fixDown(u)                  //now INV holds for u too
    }
}

Theorem (not in the text): makeHeap runs in \( O(n) \) time.

Proof. Consider the full binary tree case \( n = 2^{h+1} - 1 \). The worst possible thing that can happen is that fixDown has to swap once for every internal node just above the leaves, twice for each node in the level above, three times for the next level, up to \( h \) times for the root. Putting \( m = 2^{h-1} \approx n/4 \), this worst-case time adds up to

\[
m + \frac{m}{2} + \frac{m}{4} + \frac{m}{8} + \cdots + \frac{m}{2^h}.
\]

This is order-of \( 2m \) times the sum \( \sum_{i=1}^{h} \frac{i}{2^i} \). This sum converges even when \( h \) is allowed to go to infinity, so the time is \( O(m) = O(n) \). \( \square \)
Corollary: For any $k$, the top $k$ items in an arbitrary unsorted array of $n$ items can be found in $O(n + k \log n)$ time.

When $k = o(n)$, this beats the $\Theta(n \log n)$ time for sorting the array and outputting the top $k$ elements. Often $k$ is regarded as a constant, such as for generating "Top Ten" lists.

The case $k = n$ gives us the whole sorted array. We can get it neatly by:

```java
makeHeap(arr, n)
while(!isEmpty) {
    pop()
}
```

By the behavior of $\text{pop()}$ always putting the popped element past the end, the result of iterating it is a fully sorted array, done entirely in-place. This is HeapSort. It, too, runs in $\Theta(n \log n)$ time. The observed downside compared to QuickSort is that it tends to make more swaps on randomly generated cases of arrays. For HeapSort, one does not need $\text{makeHeap}$ to run in $O(n)$ time (so the text is OK to ignore that detail)---but it helps.

Heaps With External Updates (not in the text)

The real usefulness of coding $\text{fixDown}$ and $\text{fixUp}$ as helpers is to model dynamic updates.